



The Open University

M381 Number Theory and
Mathematical Logic



Mathematical Logic **Unit 7**

Formal Number Theory II

This system proves its own consistency

$$1+1=2$$

$\mathbb{R}$ is not countable

This computation doesn't halt

$$\forall x \forall y (x \cdot y) = (y \cdot x)$$



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INTRODUCTION

In *Unit 4* we introduced a formal language, and in *Units 5* and *6* we described a set of rules for carrying out logical deductions involving formulas of the language. In *Unit 6* we claimed (although we omitted the proof) that the Adequacy Theorem is true, that is, the rules of our formal system are adequate for deriving all logical consequences expressible in our language.

Our purpose in introducing the formal system was to tackle the questions of Leibniz and Hilbert about number theory. For the moment we confine our attention to Leibniz's Question. This was originally expressed in *Unit 1* as:

Is there an algorithm for deciding which statements of number theory are true?

We have seen that we are able to formulate this question more precisely. We interpret 'statement of number theory' to mean 'sentence of our formal language' and 'true' as meaning 'true in the standard interpretation $\mathcal{N}$ '. The theory consisting of all sentences which are true in the standard interpretation is named Complete Arithmetic (CA). We saw in *Unit 6* that we can thus rephrase Leibniz's Question as:

Is there an algorithm for deciding which sentences of our formal language are in the set CA?

Our approach to answering this question is to ask whether there is a set S of sentences simpler than CA to use as axioms for CA. As discussed in Subsection 3.2 of *Unit 6*, at the very least we need some axioms to express basic facts about addition and multiplication which are true in the standard interpretation. We gave the following criteria for a set S of sentences which we can use as axioms for number theory.

- 1 All the axioms in S should be true in the standard interpretation $\mathcal{N}$.
- 2 There should be an algorithm to determine whether a given sentence is in S .
- 3 We should be able to derive as many theorems as possible from the axioms.
- 4 The set of axioms should be as simple as possible.

To this end we began to study the theory Q obtained from just seven axioms:

- $Q1 \quad \forall x \forall y (x' = y' \rightarrow x = y)$
- $Q2 \quad \forall x \neg 0 = x'$
- $Q3 \quad \forall x (\neg x = 0 \rightarrow \exists y x = y')$
- $Q4 \quad \forall x (x + 0) = x$
- $Q5 \quad \forall x \forall y (x + y') = (x + y)'$
- $Q6 \quad \forall x (x \cdot 0) = 0$
- $Q7 \quad \forall x \forall y (x \cdot y') = ((x \cdot y) + x)$

At first sight it is very surprising that by studying Q we can throw any light on whether there is a suitable set S of axioms for CA. The axioms of Q are definitely not suitable: we did not find it difficult to find sentences in CA which are not theorems of Q . However, in this unit we shall see that appearances are deceptive. We begin, in Section 1 and Subsection 2.1, by looking at some theorems of Q . Then, in Subsection 2.2, we introduce the notion of a function being *representable* in a formal system. We shall discover that, although Q is quite a weak theory, it is sufficiently powerful to allow all total recursive functions to be representable in Q . It will turn out, in *Unit 8*, that this enables us to show that Leibniz's Question has a negative solution. This unit concludes, in Section 3, with a discussion of *diagonalization* and *Gödel's Diagonal Lemma*, which we shall need in order to answer Leibniz's Question in *Unit 8*.

We return to Hilbert's Question in *Unit 8*.

In *Unit 8*, we make this question even more precise using the terminology of recursive functions.

We would hope for a simple set S of sentences for which the theory $\text{Th}(S)$, all the sentences that we can derive using the sentences in S as assumptions, equals CA.

1 THEOREMS OF Q

In this section we look at some of the theorems of the theory Q . In

Subsection 1.1 we look at the proof of some equations and inequations in Q , while in Subsection 1.2 we look at some of the theorems which involve quantifiers.

The term *inequation* is explained below.

1.1 Fragments of arithmetic

In this subsection we look at some equations and inequations that can be derived in the theory Q . Recall that this means that they can be derived using our rules of proof from assumptions (if any are needed) included among the axioms of Q . As Q has only seven axioms, it is usually not too difficult to decide which of these axioms we are likely to need to use as assumptions. Then we can bring into play the techniques for finding formal proofs that we described in *Unit 6*.

We begin by introducing some useful notation.

- We shall sometimes use $\mathbf{n}$ as an abbreviation for the term $\mathbf{0} \overset{\text{'''...'''}}{\underbrace{\hspace{1cm}}}_{n \text{ times}}$ so that, for example, $\mathbf{2}$ stands for the term $\mathbf{0''}$. In the standard interpretation, $\mathbf{0}$ is interpreted as 0 and ' as 'add one', so that the term $\mathbf{n}$ is interpreted as the natural number n .
- We shall sometimes use $\tau_1 \neq \tau_2$, referred to as an *inequation*, as abbreviation for the formula $\neg \tau_1 = \tau_2$. This abbreviation will not be used in formal proofs.

Example 1.1

We show that $\vdash_Q \neg \mathbf{0}' = \mathbf{0''}$, that is

$$\vdash_Q \neg \mathbf{0}' = \mathbf{0''}$$

As the formula we are trying to prove involves the arithmetical symbols $\mathbf{0}$ and ' but neither of $+$ and $\cdot$, it is reasonable to guess that we won't need axioms $Q4$ to $Q7$, and hence that the only axioms of Q that we might need are $Q1$, $Q2$ and $Q3$. In fact, we only need $Q1$ and $Q2$, and we show that

$$\forall x \forall y (x' = y' \rightarrow x = y), \forall x \neg \mathbf{0} = x' \vdash \neg \mathbf{0}' = \mathbf{0''}$$

1	(1)	$\forall x \forall y (x' = y' \rightarrow x = y)$	Ass
2	(2)	$\forall x \neg \mathbf{0} = x'$	Ass
3	(3)	$\mathbf{0}' = \mathbf{0''}$	Ass
1	(4)	$\forall y (\mathbf{0}' = y' \rightarrow \mathbf{0} = y)$	UE, 1
1	(5)	$(\mathbf{0}' = \mathbf{0''} \rightarrow \mathbf{0} = \mathbf{0}')$	UE, 4
1, 3	(6)	$\mathbf{0} = \mathbf{0}'$	Taut, 3, 5
2	(7)	$\neg \mathbf{0} = \mathbf{0}'$	UE, 2
1, 2, 3	(8)	$(\mathbf{0} = \mathbf{0}' \ \& \ \neg \mathbf{0} = \mathbf{0}')$	Taut, 6, 7
1, 2	(9)	$(\mathbf{0}' = \mathbf{0''} \rightarrow (\mathbf{0} = \mathbf{0}' \ \& \ \neg \mathbf{0} = \mathbf{0}'))$	CP, 8
1, 2	(10)	$\neg \mathbf{0}' = \mathbf{0''}$	Taut, 9

Axiom $Q3$ is never used in proofs in this unit, though it does play a role in the (omitted) proof of the Representability Theorem (Theorem 2.5 of this unit).

The proof uses the techniques of Subsection 1.2 of *Unit 6*.

We begin by introducing the axioms we think we shall need to use as assumptions on lines 1 and 2. As the desired conclusion is a negation, we use technique (T4) and introduce $\mathbf{0}' = \mathbf{0''}$ as an assumption on line 3 and aim for a proof by contradiction.

We then follow (T2) by using the UE Rule to remove universal quantifiers from the assumptions. You can see that in doing this on lines 4 and 5 we have chosen to substitute the term $\mathbf{0}$ for the variable x , and $\mathbf{0}'$ for y .

There is, of course, no need to put all the axioms of Q likely to be used at the beginning of a proof. They can be introduced when and where one realizes that they are needed.

Similarly, when we applied the UE Rule to the axiom Q2 on line 7, we substituted the term $\mathbf{0}$ for the variable x . There is no difficulty with the validity of these applications of the UE Rule. The terms $\mathbf{0}$ and $\mathbf{0}'$ do not contain any variables, so they may be freely substituted for any variable in any formula.

But why did we choose to make these substitutions? Our choice to substitute $\mathbf{0}$ for x and $\mathbf{0}'$ for y was made with an eye on the formula, $\neg \mathbf{0}' = \mathbf{0}''$, that we were aiming to derive. Our strategy was based on the following idea. The axiom Q1 can be thought of as saying 'you can subtract 1 from both sides of an equation between positive numbers', and Q2 as saying that '0 is not a successor number'. From this we can deduce that 'if $1 = 2$ then $0 = 1$ ' and ' $0 \neq 1$ ', and it then follows that $1 \neq 2$. It can be seen that our formal proof follows the pattern of this informal argument.

The final line of the proof depends only on assumptions 1 and 2, which are axioms Q1 and Q2, so we have shown that $\vdash_Q \neg \mathbf{0}' = \mathbf{0}''$. ♦

Problem 1.1

Show that it is not the case that $\vdash \mathbf{1} \neq \mathbf{2}$. *Hint:* By the Correctness Theorem, you need only show that it is not the case that $\mathbf{1} \neq \mathbf{2}$ is true in every interpretation. Thus all you need do is find one interpretation in which it is not true.

Problem 1.2

(a) Show that $\vdash_Q \mathbf{2} \neq \mathbf{3}$, that is

$$\vdash_Q \neg \mathbf{0}'' = \mathbf{0}'''$$

Hint: You will need to use Q1 twice, once to get from $\mathbf{2} = \mathbf{3}$ to $\mathbf{1} = \mathbf{2}$, and then again to get from $\mathbf{1} = \mathbf{2}$ to $\mathbf{0} = \mathbf{1}$.

(b) Show that $\vdash_Q \mathbf{0} \neq \mathbf{3}$, that is

$$\vdash_Q \neg \mathbf{0} = \mathbf{0}'''$$

(c) Show that $\vdash_Q \mathbf{4} \neq \mathbf{1}$, that is,

$$\vdash_Q \neg \mathbf{0}'''' = \mathbf{0}'$$

The example and problems above suggest that Q has enough axioms for us to derive simple inequations between numerical expressions. Next we shall derive some equations between numerical expressions. The idea here is to use the recursive nature of the axioms Q4, Q5, Q6 and Q7.

Example 1.2

We show that $\vdash_Q (\mathbf{2} + \mathbf{1}) = \mathbf{3}$, that is

$$\vdash_Q (\mathbf{0}'' + \mathbf{0}') = \mathbf{0}'''$$

In fact, we show that we can derive the result using only axioms Q4 and Q5. Here is our formal proof.

1	(1)	$\forall x (x + \mathbf{0}) = x$	Ass
2	(2)	$\forall x \forall y (x + y') = (x + y)'$	Ass
2	(3)	$\forall y (\mathbf{0}'' + y') = (\mathbf{0}'' + y)'$	UE, 2
2	(4)	$(\mathbf{0}'' + \mathbf{0}') = (\mathbf{0}'' + \mathbf{0})'$	UE, 3
1	(5)	$(\mathbf{0}'' + \mathbf{0}) = \mathbf{0}''$	UE, 1
1, 2	(6)	$(\mathbf{0}'' + \mathbf{0}') = \mathbf{0}'''$	Sub, 4, 5

As mentioned in Solution 3.4 of Unit 6, these axioms express the recursive definitions of addition and multiplication.

We begin by introducing as assumptions the two axioms of Q that we think we shall need to use. We then look at the formula we are trying to derive for a clue as to which terms we should substitute in these assumptions when we use the UE Rule. We have chosen to substitute $0''$ for x and 0 for y in our assumption 2 (axiom $Q5$) because the resulting formula, on line 4 of our proof, has on the left-hand side the term $(0'' + 0')$ which also occurs on the left-hand side of the formula we are trying to derive. The formula on line 4 'reduces' $(0'' + 0')$ to $(0'' + 0)'$. We can simplify this term by substituting $0''$ for x in assumption 1 (axiom $Q4$). The resulting equation on line 5 enables us to use the Sub Rule to substitute $0''$ for $(0'' + 0)$ in the formula on line 4.

The final line of the proof depends only on assumptions 1 and 2, which are axioms $Q4$ and $Q5$, so we have shown that $\vdash_Q (0'' + 0') = 0'''$. ♦

For formulas involving larger numbers, more reductions employing the axiom $Q5$ will be needed before we can use the axiom $Q4$, as the following example shows.

Example 1.3

We show that $\vdash_Q (2 + 2) = 4$, that is

$$\vdash_Q (0'' + 0'') = 0''''$$

Here is our formal proof.

- | | | | |
|------|-----|---|-----------|
| 1 | (1) | $\forall x (x + 0) = x$ | Ass |
| 2 | (2) | $\forall x \forall y (x + y') = (x + y)'$ | Ass |
| 2 | (3) | $\forall y (0'' + y') = (0'' + y)'$ | UE, 2 |
| 2 | (4) | $(0'' + 0'') = (0'' + 0')'$ | UE, 3 |
| 2 | (5) | $(0'' + 0') = (0'' + 0)'$ | UE, 3 |
| 2 | (6) | $(0'' + 0'') = (0'' + 0)''$ | Sub, 4, 5 |
| 1 | (7) | $(0'' + 0) = 0''$ | UE, 1 |
| 1, 2 | (8) | $(0'' + 0'') = 0''''$ | Sub, 6, 7 |

The proof begins, as in Example 1.2, with the introduction, as assumptions, of the axioms $Q4$ and $Q5$. Since the final line depends only on these assumptions, this proof shows that $\vdash_Q (0'' + 0'') = 0''''$.

Note the reductions used in this proof. Line 4 expresses $(2 + 2)$ in terms of $(2 + 1)$, and line 5 expresses $(2 + 1)$ in terms of $(2 + 0)$. Both of these reductions use axiom $Q5$. Then axiom $Q4$ is used on line 7 to equate $(2 + 0)$ with 2 . Lines 6 and 8 use the Sub Rule to 'substitute equals for equals' and thus enable us to derive finally the formula $(0'' + 0'') = 0''''$. ♦

Problem 1.3

Show that $\vdash_Q (0 + 1) = 1$, that is

$$\vdash_Q (0 + 0') = 0'$$

Proofs of formulas similar to those above, but involving equations between terms which include the multiplication symbol, are necessarily more complicated. This is because, in reducing a term using axiom $Q7$, an addition symbol is introduced, and we need to eliminate this using axioms $Q4$ and $Q5$ before we can make further reductions using axiom $Q7$. Here is an example of such a proof.

Example 1.4

We show that $\vdash_Q (1 \cdot 2) = 2$, that is

$$\vdash_Q (0' \cdot 0'') = 0''$$

1	(1)	$\forall x (x + 0) = x$	Ass
2	(2)	$\forall x \forall y (x + y') = (x + y)'$	Ass
3	(3)	$\forall x (x \cdot 0) = 0$	Ass
4	(4)	$\forall x \forall y (x \cdot y') = ((x \cdot y) + x)$	Ass
4	(5)	$\forall y (0' \cdot y') = ((0' \cdot y) + 0')$	UE, 4
4	(6)	$(0' \cdot 0'') = ((0' \cdot 0') + 0')$	UE, 5
2	(7)	$\forall y ((0' \cdot 0') + y') = ((0' \cdot 0') + y)'$	UE, 2
2	(8)	$((0' \cdot 0') + 0') = ((0' \cdot 0') + 0)'$	UE, 7
1	(9)	$((0' \cdot 0') + 0) = (0' \cdot 0')$	UE, 1
1, 2	(10)	$((0' \cdot 0') + 0') = (0' \cdot 0')'$	Sub, 8, 9
1, 2, 4	(11)	$(0' \cdot 0'') = (0' \cdot 0')'$	Sub, 6, 10
4	(12)	$(0' \cdot 0') = ((0' \cdot 0) + 0')$	UE, 5
2	(13)	$\forall y ((0' \cdot 0) + y') = ((0' \cdot 0) + y)'$	UE, 2
2	(14)	$((0' \cdot 0) + 0') = ((0' \cdot 0) + 0)'$	UE, 13
1	(15)	$((0' \cdot 0) + 0) = (0' \cdot 0)$	UE, 1
1, 2	(16)	$((0' \cdot 0) + 0') = (0' \cdot 0)'$	Sub, 14, 15
1, 2, 4	(17)	$(0' \cdot 0') = (0' \cdot 0)'$	Sub, 12, 16
1, 2, 4	(18)	$(0' \cdot 0'') = (0' \cdot 0)''$	Sub, 11, 17
3	(19)	$(0' \cdot 0) = 0$	UE, 3
1, 2, 3, 4	(20)	$(0' \cdot 0'') = 0''$	Sub, 18, 19

Since the assumptions that we have used to derive the formula on line 20 are axioms $Q4$, $Q5$, $Q6$ and $Q7$, this formal proof does indeed show that $\vdash_Q (1 \cdot 2) = 2$.

This proof looks rather long and complicated, but it can be broken down into a few basic steps. The length arises because we need to introduce four axioms as assumptions, and then use the UE Rule to obtain the particular instances of these axioms needed to derive our desired conclusion. Most of the lines are taken up in this way.

On line 6 we make the obvious reduction of the term $(0' \cdot 0'')$ using the axiom $Q7$, but this introduces an addition symbol. We eliminate this symbol on lines 7 to 11 so that on line 11 we have the term $(1 \cdot 2)$ expressed in terms of $(1 \cdot 1)$. Next we repeat this procedure with the term $(1 \cdot 1)$ and express it in terms of $(1 \cdot 0)$ on line 17. We can then complete the proof using axiom $Q6$ (the first recursion equation for multiplication) which tells us that $(1 \cdot 0) = 0$. ♦

Problem 1.4

Show that $\vdash_Q (1 \cdot 1) = 1$, that is

$$\vdash_Q (0' \cdot 0') = 0'$$

The next problem involves both an inequation and an addition. So you should expect to use axioms $Q1$, $Q2$ and also axioms $Q4$, $Q5$.

Problem 1.5

Show that $\vdash_Q (1 + 1) \neq 1$, that is

$$\vdash_Q \neg (0' + 0') = 0'$$

1.2 Theorems involving quantifiers

We continue this section by presenting some formal proofs of theorems of Q which involve quantifiers. We use the techniques for finding formal proofs set out in Subsection 1.2 of *Unit 6*. We normally start each formal proof by introducing as assumptions those axioms of Q that we think we are going to need.

Example 1.5

We show that

$$\vdash_Q \forall x \exists y (x + y) = x$$

Recall that our standard strategy for deriving a universal formula of the form $\forall x \phi$ is to aim to derive the formula ϕ and then use the UI Rule. So here we aim first to derive the formula $\exists y (x + y) = x$. Since this is an existential formula, our standard strategy suggests that we aim to derive the formula $(x + \tau) = x$ for some term τ . Which τ should we take? The term τ we choose should ideally be such that $(x + \tau) = x$ is true in all interpretations of Q , in particular in the standard interpretation $\mathcal{N}$. Therefore, we need τ such that, for any value in $\mathbb{N}$ (the domain of $\mathcal{N}$) given to x , $(x + \tau) = x$ is true in $\mathcal{N}$. The obvious choice for τ is the term 0 . For this choice of τ , we see that the formula $(x + 0) = x$ can be derived from axiom $Q4$ by a single use of the UE Rule. We thus arrive at the following formal proof.

- 1 (1) $\forall x (x + 0) = x$ Ass
- 1 (2) $(x + 0) = x$ UE, 1
- 1 (3) $\exists y (x + y) = x$ EI, 2
- 1 (4) $\forall x \exists y (x + y) = x$ UI, 3

As assumption 1 is the axiom $Q4$, we have shown that

$$\vdash_Q \forall x \exists y (x + y) = x. \quad \blacklozenge$$

Problem 1.6

Show that

$$\vdash_Q \forall x \exists y (x' + 0) = y'$$

Example 1.6

We show that

$$\vdash_Q \forall x ((0 \cdot 0) \cdot x) = ((0 \cdot x) + (x \cdot 0))$$

The sentence $\forall x ((0 \cdot 0) \cdot x) = ((0 \cdot x) + (x \cdot 0))$ is certainly true in the standard interpretation $\mathcal{N}$ because, however we interpret x , each of the terms in the atomic formula $((0 \cdot 0) \cdot x) = ((0 \cdot x) + (x \cdot 0))$ will be interpreted as the number 0. Nevertheless, since we noted in *Unit 6*, Subsection 3.2, that the sentence $\forall x (0 \cdot x) = 0$ is *not* a theorem of Q , we might feel dubious as to whether the sentence

$\forall x ((0 \cdot 0) \cdot x) = ((0 \cdot x) + (x \cdot 0))$ is a theorem of Q . However, we can use axiom $Q6$ to ‘simplify’ $((0 \cdot 0) \cdot x)$ to $(0 \cdot x)$, and axioms $Q6$ and $Q4$ successively to ‘simplify’ $((0 \cdot x) + (x \cdot 0))$ to $((0 \cdot x) + 0)$ and then to $(0 \cdot x)$. Thus the atomic formula $((0 \cdot 0) \cdot x) = ((0 \cdot x) + (x \cdot 0))$ may be derived by using the Sub Rule. We can then use the UI Rule to derive the sentence $\forall x ((0 \cdot 0) \cdot x) = ((0 \cdot x) + (x \cdot 0))$.

We cannot ‘simplify’ the term $(0 \cdot x)$ any further using solely the axioms of Q .

The details of this strategy are embedded in the following formal proof, which we start by introducing as assumptions the two axioms of Q that we need to use.

1	(1)	$\forall x (x + \mathbf{0}) = x$	Ass
2	(2)	$\forall x (x \cdot \mathbf{0}) = \mathbf{0}$	Ass
	(3)	$((\mathbf{0} \cdot \mathbf{0}) \cdot x) = ((\mathbf{0} \cdot \mathbf{0}) \cdot x)$	II
2	(4)	$(\mathbf{0} \cdot \mathbf{0}) = \mathbf{0}$	UE, 2
2	(5)	$((\mathbf{0} \cdot \mathbf{0}) \cdot x) = (\mathbf{0} \cdot x)$	Sub, 3, 4
	(6)	$((\mathbf{0} \cdot x) + (x \cdot \mathbf{0})) = ((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$	II
2	(7)	$(x \cdot \mathbf{0}) = \mathbf{0}$	UE, 2
2	(8)	$((\mathbf{0} \cdot x) + \mathbf{0}) = ((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$	Sub, 6, 7
1	(9)	$((\mathbf{0} \cdot x) + \mathbf{0}) = (\mathbf{0} \cdot x)$	UE, 1
1, 2	(10)	$(\mathbf{0} \cdot x) = ((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$	Sub, 8, 9
1, 2	(11)	$((\mathbf{0} \cdot \mathbf{0}) \cdot x) = ((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$	Sub, 5, 10
1, 2	(12)	$\forall x ((\mathbf{0} \cdot \mathbf{0}) \cdot x) = ((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$	UI, 11

Note that, when trying to show that a term τ such as $((\mathbf{0} \cdot \mathbf{0}) \cdot x)$ or $((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$ equals something simpler, we need to use the II Rule to introduce the formula $\tau = \tau$, as on lines 3 and 6, before proceeding with the simplification.

Since assumptions 1 and 2 are axioms $Q4$ and $Q6$, we have shown that $\vdash_Q \forall x ((\mathbf{0} \cdot \mathbf{0}) \cdot x) = ((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$.

Notice that, in performing the simplifications, we kept the term $((\mathbf{0} \cdot \mathbf{0}) \cdot x)$ on the left-hand side of the equation in deriving line 5 and the term $((\mathbf{0} \cdot x) + (x \cdot \mathbf{0}))$ on the right-hand side of the equation in deriving line 10, to facilitate the final substitution on line 11. There are other ways of facilitating the final substitution, but the technique here generalizes more readily to more complicated cases. However, what you might regard as the most obvious simplification technique, namely to simplify the right-hand side in each case, to give $((\mathbf{0} \cdot \mathbf{0}) \cdot x) = (\mathbf{0} \cdot x)$ on line 5 and $((\mathbf{0} \cdot x) + (x \cdot \mathbf{0})) = (\mathbf{0} \cdot x)$ on line 10, does not work well; this is because we cannot then apply the Sub Rule directly to these lines to obtain line 11 since $(\mathbf{0} \cdot x)$ appears on the right-hand side in each case. ♦

Problem 1.7

Show that

$$\vdash_Q \forall x (\mathbf{0} \cdot (x + \mathbf{0})) = ((\mathbf{0} \cdot x) + (\mathbf{0} \cdot \mathbf{0}))$$

Example 1.7

We show that

$$\vdash_Q \exists y ((y + y) \cdot y) = y'$$

Following technique (T10) for proving existential formulas, we aim to derive the sentence $\exists y ((y + y) \cdot y) = y'$ from a formula of the form $((\tau + \tau) \cdot \tau) = \tau'$ where τ is some specific term. We need τ such that $((\tau + \tau) \cdot \tau) = \tau'$ is true in the standard interpretation $\mathcal{N}$. As the only solution in the natural numbers to $(y + y) \cdot y = y + 1$ is $y = 1$, the obvious choice for τ is the term $\mathbf{0}'$. So our aim is first to derive the sentence $((\mathbf{0}' + \mathbf{0}') \cdot \mathbf{0}') = \mathbf{0}''$. As you can imagine, we shall need axioms $Q4$ to $Q7$. We first use $Q4$ and $Q5$ to derive $((\mathbf{0}' + \mathbf{0}') \cdot \mathbf{0}') = (\mathbf{0}'' \cdot \mathbf{0}')$, then $Q6$ and $Q7$ to transform the right-hand side into $(\mathbf{0} + \mathbf{0}'')$, and then $Q4$ and $Q5$ again to transform it into $\mathbf{0}''$.

In standard algebraic notation, the equation is $2y^2 = y + 1$ with solutions $y = 1, -\frac{1}{2}$. So 1 is the only solution in the natural numbers.

1	(1)	$\forall x (x + 0) = x$	Ass
2	(2)	$\forall x \forall y (x + y') = (x + y)'$	Ass
3	(3)	$\forall x (x \cdot 0) = 0$	Ass
4	(4)	$\forall x \forall y (x \cdot y') = ((x \cdot y) + x)$	Ass
	(5)	$((0' + 0') \cdot 0') = ((0' + 0') \cdot 0')$	II
2	(6)	$\forall y (0' + y') = (0' + y)'$	UE, 2
2	(7)	$(0' + 0') = (0' + 0)'$	UE, 6
1	(8)	$(0' + 0) = 0'$	UE, 1
1, 2	(9)	$(0' + 0') = 0''$	Sub, 7, 8
1, 2	(10)	$((0' + 0') \cdot 0') = (0'' \cdot 0')$	Sub, 5, 9
4	(11)	$\forall y (0'' \cdot y') = ((0'' \cdot y) + 0'')$	UE, 4
4	(12)	$(0'' \cdot 0') = ((0'' \cdot 0) + 0'')$	UE, 11
3	(13)	$(0'' \cdot 0) = 0$	UE, 3
3, 4	(14)	$(0'' \cdot 0') = (0 + 0'')$	Sub, 12, 13
1, 2, 3, 4	(15)	$((0' + 0') \cdot 0') = (0 + 0'')$	Sub, 10, 14
2	(16)	$\forall y (0 + y') = (0 + y)'$	UE, 2
2	(17)	$(0 + 0'') = (0 + 0')'$	UE, 16
2	(18)	$(0 + 0') = (0 + 0)'$	UE, 16
1	(19)	$(0 + 0) = 0$	UE, 1
1, 2	(20)	$(0 + 0') = 0'$	Sub, 18, 19
1, 2	(21)	$(0 + 0'') = 0''$	Sub, 17, 20
1, 2, 3, 4	(22)	$((0' + 0') \cdot 0') = 0''$	Sub, 15, 21
1, 2, 3, 4	(23)	$\exists y ((y + y) \cdot y) = y'$	EI, 22

Since assumptions 1, 2, 3 and 4 are, respectively, axioms $Q4$, $Q5$, $Q6$ and $Q7$, we have shown that $\vdash_Q \exists y ((y + y) \cdot y) = y'$. $\blacklozenge$

Problem 1.8

Show that

$$\vdash_Q \exists y (y' \cdot y) = (y + 0')$$

We formulated the axioms of Q using the specific variables x and y . As the interpretation of a sentence is unchanged if the variables are renamed, it should not be a surprise that if we take any axiom of Q and replace x and y by any other two variables, then the resulting sentence is also a theorem of Q . We illustrate this in the next example.

Example 1.8

We show that

$$\vdash_Q \forall y \forall x (y + x') = (y + x)'$$

We have the following formal proof.

1	(1)	$\forall x \forall y (x + y') = (x + y)'$	Ass
1	(2)	$\forall y (z + y') = (z + y)'$	UE, 1
1	(3)	$(z + x') = (z + x)'$	UE, 2
1	(4)	$\forall z (z + x') = (z + x)'$	UI, 3
1	(5)	$(y + x') = (y + x)'$	UE, 4
1	(6)	$\forall x (y + x') = (y + x)'$	UI, 5
1	(7)	$\forall y \forall x (y + x') = (y + x)'$	UI, 6

Since assumption 1 is axiom $Q5$, we have shown that $\vdash_Q \forall y \forall x (y + x') = (y + x)'$. $\blacklozenge$

Note that the sentence $\forall y \forall x (y + x') = (y + x)'$ is obtained from axiom $Q5$ by replacing the variable x by y and the variable y by x .

Note that when we apply the UE Rule to the formula on line 1, we need to substitute for the occurrences of x in $\forall y (x + y') = (x + y)'$ a variable which is distinct from both x and y . We choose to use z , but any variable other than x and y would have done.

Problem 1.9

Show that

$$\vdash_Q \forall y \forall x (y \cdot x') = ((y \cdot x) + y)$$

Problem 1.10

One of the following sentences is a theorem of Q and one is not. Give a formal proof of the sentence that is a theorem of Q and explain why the other sentence is not a theorem of Q .

(a) $\forall x (x + 0') = x'$

(b) $\forall x (x \cdot 0') = x$

2 REPRESENTABILITY

In Unit 6, Subsection 3.2, we gave several examples of sentences that are true in the standard interpretation but which are not derivable in Q , i.e. which are not derivable from the axioms of Q . For example, we saw that the sentence $\forall x \forall y (x + y) = (y + x)$, which expresses the fact that addition is commutative, is not derivable in Q . Examples such as this may have led you to believe that the theory Q is so weak as to be useless for any serious purpose. In this section we discuss an important theorem, the Representability Theorem, which should dispel this impression. This theorem connects total functions $f: \mathbb{N}^n \rightarrow \mathbb{N}$ which are *representable* in a theory such as Q with the total recursive functions discussed in Units 1 to 3. The fact that this theorem holds for Q but not for any significantly weaker theory explains why we chose to study the theory Q . We shall explain what is meant by a representable function in Subsection 2.2, after some preliminary work in Subsection 2.1.

This key theorem plays a significant role in Unit 8.

2.1 More theorems of Q

In Subsection 1.1 we established some arithmetical equations and inequations as theorems of Q . For example, we showed the following.

$$\vdash_Q 1 \neq 2$$

$$\vdash_Q (2 + 1) = 3$$

$$\vdash_Q (1 \cdot 2) = 2$$

These are particular cases of more general results, some of which will now be established.

Theorem 2.1

For all natural numbers i and j :

(a) if $i = j$, then $\vdash_Q i = j$;

(b) if $i \neq j$, then $\vdash_Q i \neq j$.

Proof

This theorem asserts that infinitely many different sentences are derivable in the theory Q . We therefore cannot justify this claim by presenting a separate formal proof of each of these sentences. What we do instead is to give general arguments to show that the required formal proofs exist.

- (a) Let i and j be natural numbers and suppose that $i = j$. Then the formula $\mathbf{i} = \mathbf{j}$ is identical to the formula $\mathbf{i} = \mathbf{i}$ which can be derived in one line using the II Rule and no assumptions. To be specific, for each natural number i , a formal proof is

$$(1) \quad \mathbf{i} = \mathbf{i} \quad \text{II}$$

Hence, if $i = j$, then $\vdash_Q \mathbf{i} = \mathbf{j}$.

- (b) Now let i and j be natural numbers and suppose that $i \neq j$.

We consider first the case where $i = 0$. Then $j > 0$ and so $j = k + 1$ for some natural number k . Hence in this case the formula $\mathbf{i} \neq \mathbf{j}$ is $\neg \mathbf{0} = \mathbf{k}'$. We can easily give a formal proof of such a formula using axiom Q2: for each natural number k ,

$$1 \quad (1) \quad \forall x \neg \mathbf{0} = x' \quad \text{Ass}$$

$$1 \quad (2) \quad \neg \mathbf{0} = \mathbf{k}' \quad \text{UE, 1}$$

is such a formal proof. The only assumption is axiom Q2. It follows that $\vdash_Q \neg \mathbf{0} = \mathbf{k}'$. Thus, in the case where $i = 0$, we have $\vdash_Q \mathbf{i} \neq \mathbf{j}$.

Next we consider the case where $i > 0$.

We assume first that $i < j$. The sentence $\mathbf{i} \neq \mathbf{j}$ is

$$\neg \underbrace{\mathbf{0}''''\dots'}_{i \text{ times}} = \underbrace{\mathbf{0}''''\dots'}_{j \text{ times}}$$

Since this sentence, which we are aiming to show can be derived in the theory Q , is a negation, our strategy will be to assume the sentence $\underbrace{\mathbf{0}''''\dots'}_{i \text{ times}} = \underbrace{\mathbf{0}''''\dots'}_{j \text{ times}}$ and then aim to derive a contradiction.

Repeated use of axiom Q1 enables us to drop one successor symbol from the left- and right-hand sides of the formula $\underbrace{\mathbf{0}''''\dots'}_{i \text{ times}} = \underbrace{\mathbf{0}''''\dots'}_{j \text{ times}}$ at each

step until, after i such steps, we end up with the sentence $\mathbf{0} = \underbrace{\mathbf{0}''''\dots'}_{j-i \text{ times}}$.

Note that, as $j > i$, we have $j - i > 0$, so that this is a sentence of the form $\mathbf{0} = \mathbf{k}'$, where $k = j - i - 1 \geq 0$. Then a use of axiom Q2 enables us to derive a contradiction.

In our formal proof, it will be convenient to use some new notation, namely $[\mathbf{n} - \mathbf{m}]$, which is defined if $n \geq m$ and stands for $\underbrace{\mathbf{0}''''\dots'}_{n-m \text{ times}}$

(so that, in particular, if $n = m$ then $[\mathbf{n} - \mathbf{m}] = \mathbf{0}$). Therefore, for example, $[\mathbf{i} - \mathbf{1}]$ stands for $\underbrace{\mathbf{0}''''\dots'}_{i-1 \text{ times}}$, $[\mathbf{i} - \mathbf{2}]$ stands for $\underbrace{\mathbf{0}''''\dots'}_{i-2 \text{ times}}$, $[\mathbf{i} - \mathbf{1}]'$

stands for $\underbrace{\mathbf{0}''''\dots'}_{i \text{ times}}$, and so is the same as the term $\mathbf{i}$, and $[\mathbf{i} - \mathbf{2}]'$ is the same as the term $[\mathbf{i} - \mathbf{1}]$.

In this case, since no axiom of Q needs to be used in the proof, we have not only $\vdash_Q \mathbf{i} = \mathbf{j}$ but also $\vdash \mathbf{i} = \mathbf{j}$.

Thus, in the case where $0 < i < j$, we have a formal proof of the following form.

1	(1)	$i = j$	Ass
2	(2)	$\forall x \forall y (x' = y' \rightarrow x = y)$	Ass
2	(3)	$\forall y ([i - 1]' = y' \rightarrow [i - 1] = y)$	UE, 2
2	(4)	$([i - 1]' = [j - 1]' \rightarrow [i - 1] = [j - 1])$	UE, 3
1, 2	(5)	$[i - 1] = [j - 1]$	Taut, 1, 4
	$\vdots$		
1, 2	(8)	$[i - 2] = [j - 2]$	Taut, 5, 7
	$\vdots$		
1, 2	$(3i + 2)$	$0 = k'$	Taut, $3i - 1, 3i + 1$
$3i + 3$	$(3i + 3)$	$\forall x \neg 0 = x'$	Ass
$3i + 3$	$(3i + 4)$	$\neg 0 = k'$	UE, $3i + 3$
1, 2, $3i + 3$	$(3i + 5)$	$(0 = k' \& \neg 0 = k')$	Taut, $3i + 2, 3i + 4$
2, $3i + 3$	$(3i + 6)$	$(i = j \rightarrow (0 = k' \& \neg 0 = k'))$	CP, $3i + 5$
2, $3i + 3$	$(3i + 7)$	$\neg i = j$	Taut, $3i + 6$

Since assumptions 2 and $3i + 3$ are axioms $Q1$ and $Q2$, it follows that, in this case, $\vdash_Q i \neq j$.

In the case where $j < i$, the argument is only slightly different. We can begin a formal proof as follows.

1	(1)	$i = j$	Ass
	(2)	$i = i$	II
1	(3)	$j = i$	Sub, 1, 2

This copes with the awkwardness of axiom $Q2$ being $\forall x \neg 0 = x'$ rather than $\forall x \neg x' = 0$.

These three lines replace line 1 in the formal proof in the case where $i < j$. If $j = 0$, so that $i = k + 1$ for some natural number k , $j = i$ in the new line 3 is the same as $0 = k'$, and the proof is then completed with the last five lines of the proof in the case where $i < j$ (suitably renumbered). If $j > 0$, we continue after the three new lines above with lines 2, 3, ... of the formal proof in the case where $i < j$, renumbered as lines 4, 5, ..., with the roles of i and j interchanged, until we reach $0 = k'$. The proof is then completed exactly in the case where $i < j$ except for the renumbering of the lines. Hence in this case also we have $\vdash_Q i \neq j$. ■

Next we have a general result for addition. It again asserts that infinitely many sentences of a certain form are derivable in the theory Q . Thus our proof consists of general arguments to show that formal proofs of all these sentences exist.

Theorem 2.2

For all natural numbers i, j and k such that $i + j = k$:

- (a) $\vdash_Q (i + j) = k$
- (b) $\vdash_Q \forall x ((i + j) = x \rightarrow x = k)$

Proof

- (a) Here we use proof by Mathematical Induction. We shall prove that the proposition $P(j)$ that 'for all natural numbers i and k such that $i + j = k$, the sentence $(\mathbf{i} + \mathbf{j}) = \mathbf{k}$ can be derived from axioms $Q4$ and $Q5$ ' is true for all natural numbers j . Since we wish to prove this for all $j \geq 0$, the basis for the induction is the case where $j = 0$.

Mathematical Induction is discussed in Number Theory, Unit 1, Section 3.

So suppose that i and k are natural numbers such that $i + 0 = k$. This means that $i = k$ and hence the sentence $(\mathbf{i} + \mathbf{j}) = \mathbf{k}$ is $(\mathbf{i} + \mathbf{0}) = \mathbf{i}$. This sentence has the following very straightforward formal proof.

$$\begin{array}{ll} 1 & (1) \quad \forall x (x + \mathbf{0}) = x \quad \text{Ass} \\ 1 & (2) \quad (\mathbf{i} + \mathbf{0}) = \mathbf{i} \quad \text{UE, 1} \end{array}$$

Assumption 1 is axiom $Q4$. Hence we have shown that, in the case where $j = 0$ and $i + j = k$, the sentence $(\mathbf{i} + \mathbf{j}) = \mathbf{k}$ can be derived using at most $Q4$ and $Q5$ as assumptions. Thus we have established the basis for the induction.

We now come to the induction step. We show that, from the assumption that the result holds for $j = n$, we can deduce that it holds also for $j = n + 1$. So we assume that, whenever i and k are natural numbers such that $i + n = k$, there is a formal proof of the sentence $(\mathbf{i} + \mathbf{n}) = \mathbf{k}$ from axioms $Q4$ and $Q5$.

Now let i and k be natural numbers such that $i + (n + 1) = k$, so that $k > 0$. Then $i + n = k - 1$ and hence, by our induction hypothesis, we have a formal proof of the following form.

$$\begin{array}{ll} 1 & (1) \quad \forall x (x + \mathbf{0}) = x \quad \text{Ass} \\ 2 & (2) \quad \forall x \forall y (x + y') = (x + y)' \quad \text{Ass} \\ & \vdots \\ 1, 2 & (s) \quad (\mathbf{i} + \mathbf{n}) = [\mathbf{k} - \mathbf{1}] \quad \dots \end{array}$$

We may now continue this formal proof as follows.

$$\begin{array}{ll} 2 & (s + 1) \quad \forall y (\mathbf{i} + y') = (\mathbf{i} + y)' \quad \text{UE, 2} \\ 2 & (s + 2) \quad (\mathbf{i} + \mathbf{n}') = (\mathbf{i} + \mathbf{n})' \quad \text{UE, } s + 1 \\ 1, 2 & (s + 3) \quad (\mathbf{i} + \mathbf{n}') = [\mathbf{k} - \mathbf{1}]' \quad \text{Sub, } s, s + 2 \end{array}$$

Now $\mathbf{n}'$ is $(\mathbf{n} + \mathbf{1})$ and $[\mathbf{k} - \mathbf{1}]'$ is the term $\mathbf{k}$. Thus we have deduced that the sentence $(\mathbf{i} + (\mathbf{n} + \mathbf{1})) = \mathbf{k}$ can be derived from axioms $Q4$ and $Q5$. Hence the result holds also for $j = n + 1$.

This completes our proof by Mathematical Induction. It follows that, for all natural numbers i, j and k such that $i + j = k$, $\vdash_Q (\mathbf{i} + \mathbf{j}) = \mathbf{k}$.

- (b) Again we suppose i, j and k are natural numbers such that $i + j = k$. By part (a), it follows that there is a formal proof of the following form

$$\begin{array}{ll} 1 & (1) \quad \forall x (x + \mathbf{0}) = x \quad \text{Ass} \\ 2 & (2) \quad \forall x \forall y (x + y') = (x + y)' \quad \text{Ass} \\ & \vdots \\ 1, 2 & (s) \quad (\mathbf{i} + \mathbf{j}) = \mathbf{k} \quad \dots \end{array}$$

We may add to this formal proof the following four lines.

$$\begin{array}{ll} s + 1 & (s + 1) \quad (\mathbf{i} + \mathbf{j}) = x \quad \text{Ass} \\ 1, 2, s + 1 & (s + 2) \quad x = \mathbf{k} \quad \text{Sub, } s, s + 1 \\ 1, 2 & (s + 3) \quad ((\mathbf{i} + \mathbf{j}) = x \rightarrow x = \mathbf{k}) \quad \text{CP, } s + 2 \\ 1, 2 & (s + 4) \quad \forall x ((\mathbf{i} + \mathbf{j}) = x \rightarrow x = \mathbf{k}) \quad \text{UI, } s + 3 \end{array}$$

We may thus deduce that $\vdash_Q \forall x ((\mathbf{i} + \mathbf{j}) = x \rightarrow x = \mathbf{k})$. ■

Problem 2.1

Show that, for all natural numbers i and j ,

$$\vdash_Q (i + j) = (j + i)$$

Example 2.1

We show that if, i, j and n are natural numbers such that $i + j \neq n$ then

$$\vdash_Q (i + j) \neq n$$

Suppose i, j and n are natural numbers such that $i + j \neq n$. Let $i + j = k$, so that $k \neq n$. Since $k \neq n$, it follows from Theorem 2.1(b) that

$$\vdash_Q k \neq n$$

Since $i + j = k$, it follows from Theorem 2.2(a) that

$$\vdash_Q (i + j) = k$$

From this it follows that

$$\vdash_Q k = (i + j)$$

Hence, using the Sub Rule to replace the term k by the term $(i + j)$ in the sentence $k \neq n$, we can deduce that

$$\vdash_Q (i + j) \neq n$$

We saw in Subsection 2.1 of Unit 6 that, given $\tau_1 = \tau_2$ for any terms τ_1 and τ_2 , we can always derive $\tau_2 = \tau_1$.

There is an analogous result to Theorem 2.2 in the case of multiplication, which we state without proof.

Theorem 2.3

For all natural numbers i, j and k such that $i \times j = k$:

$$(a) \vdash_Q (i \cdot j) = k$$

$$(b) \vdash_Q \forall x ((i \cdot j) = x \rightarrow x = k)$$

This theorem can be proved using similar methods to those used in our proof of Theorem 2.2, but, because it deals with multiplication, the proof is a little more complicated.

Problem 2.2

Show that, for all natural numbers i and j ,

$$\vdash_Q (i \cdot j) = (j \cdot i)$$

We conclude this subsection with some remarks about what the theorems we have just proved imply about interpretations of the theory Q . Recall that an interpretation of Q is one in which all the theorems of Q are true. In particular all the sentences specified in Theorems 2.1 to 2.3, in Example 2.1 and in Problems 2.1 and 2.2 will be true in such an interpretation.

Suppose now that we have a non-standard interpretation of Q . The domain of this interpretation must include, for each natural number i , an interpretation of the term i . We denote the element of the domain which is the interpretation of i by i^* . By Theorem 2.1(a), if i and j are distinct natural numbers, then in the non-standard interpretation $i^* \neq j^*$, and it follows that its domain is an infinite set. The non-standard interpretation must also include operations corresponding to the addition and multiplication symbols. We use $+_*$ and $\cdot_*$ for these operations. It follows from Theorems 2.2 and 2.3 that, for all natural numbers i, j and k ,

$$i^* +_* j^* = k^* \iff i + j = k,$$

$$i^* \cdot_* j^* = k^* \iff i \cdot j = k.$$

It will also follow, from Problem 2.4 later in this section, that

$$(i^*)' = j^* \iff i' = j.$$

Thus the part of our non-standard interpretation consisting of the interpretations of the numerals $\mathbf{i}$ looks just like the standard interpretation. So any non-standard interpretation of Q essentially consists of the standard interpretation with some additional elements in its domain and corresponding additional interpretations of the addition, multiplication and successor operations when applied to these additional symbols. Recall that, in Examples 3.5 and 3.6 of *Unit 4*, we have two examples of non-standard interpretations of our formal language. The first consisted of the standard interpretation with an additional element ω , and the second consisted of the standard interpretation with two additional elements α and β . It turns out that both these non-standard interpretations of our formal language are non-standard interpretations of Q .

Notice also what the results of Problems 2.1 and 2.2 tell us. They imply that in any non-standard interpretation the operations of ‘addition’ and ‘multiplication’ which interpret the symbols $+$ and $\cdot$ must be commutative as far as the interpretations of the numerals $\mathbf{i}$ are concerned. However, as we have already noted in *Unit 6*, these operations do not need to be commutative over the whole of the domain. Thus, for example, in the non-standard interpretation of Example 3.6 of *Unit 4*, neither ‘addition’ nor ‘multiplication’ is commutative. However, to show this we must use the non-standard elements of the domain: for example, $\alpha + \beta \neq \beta + \alpha$ and $\alpha \cdot 0 \neq 0 \cdot \alpha$ are examples which show that neither ‘addition’ nor ‘multiplication’ is commutative.

In *Unit 6* we saw that neither $\forall x \forall y (x + y) = (y + x)$ nor $\forall x \forall y (x \cdot y) = (y \cdot x)$ is derivable in Q .

2.2 The Representability Theorem

In this subsection we discuss a generalization of Theorems 2.2 and 2.3 to functions other than addition and multiplication. In order to give the key definition, we introduce some new notation.

Let ϕ be a formula in which no variables other than $x_1, x_2, \dots, x_n$ have free occurrences, and let $\tau_1, \tau_2, \dots, \tau_n$ be terms. We write $\phi(\tau_1, \tau_2, \dots, \tau_n)$ for the formula which is obtained from ϕ by replacing all free occurrences of the variable x_r by the term τ_r , for $r = 1, 2, \dots, n$.

Example 2.2

- Let ϕ be the formula $(x_1 + x_2) = x_3$. Then $\phi(\mathbf{2}, \mathbf{3}, \mathbf{5})$ is the formula $(\mathbf{2} + \mathbf{3}) = \mathbf{5}$ and $\phi(\mathbf{2}, \mathbf{3}, \mathbf{7})$ is the formula $(\mathbf{2} + \mathbf{3}) = \mathbf{7}$.
- Let ϕ be the formula $(\exists x_2 x_1 = (x_2 + x_2) \ \& \ x_2 = \mathbf{0})$. Then $\phi(\mathbf{2}, \mathbf{3})$ is the formula $(\exists x_2 \mathbf{2} = (x_2 + x_2) \ \& \ \mathbf{3} = \mathbf{0})$. Here we have replaced the only occurrence of x_1 , which is a free occurrence, by the term $\mathbf{2}$. We have also replaced the only free occurrence of x_2 by the term $\mathbf{3}$. The first three occurrences of x_2 in the formula ϕ are not free occurrences and so are not replaced. $\blacklozenge$

In describing the notation $\phi(\tau_1, \tau_2, \dots, \tau_n)$ we have allowed $\tau_1, \tau_2, \dots, \tau_n$ to be any terms. In practice, in most cases they will turn out to be terms of the form $\mathbf{m}$, where m is a natural number, or a variable x_r . It is commonly the case that τ_n is the variable x_n , in which case we write $\phi(\tau_1, \tau_2, \dots, \tau_{n-1}, x_n)$ for the formula obtained from ϕ by replacing the free occurrences of the variables $x_1, x_2, \dots, x_{n-1}$ by the terms $\tau_1, \tau_2, \dots, \tau_{n-1}$, respectively, but leaving occurrences of the variable x_n unchanged.

Example 2.3

- (a) Let ϕ be the formula $(x_1 + x_2) = x_3$. Then $\phi(2, 3, x_3)$ is the formula $(2 + 3) = x_3$.
- (b) Let ϕ be the formula $(\exists x_2 x_1 = (x_2 + x_2) \ \& \ x_2 = 0)$. Then $\phi(4, x_2)$ is $(\exists x_2 4 = (x_2 + x_2) \ \& \ x_2 = 0)$. ♦

Problem 2.3

- (a) Let ϕ be the formula $((x_1 \cdot x_2) = x_3 \vee x_3 = x_2)$. Write down the formulas $\phi(3, 4, 5)$ and $\phi(1, 2, x_3)$.
- (b) Let ϕ be the formula $(\exists x_4 (x_2 \cdot (x_1 \cdot x_3)) = x_4 \rightarrow x_4 = x_3)$. Write down the formulas $\phi(1, 2, 3, 4)$ and $\phi(1, 2, 3, x_4)$.

We can now restate Theorem 2.2 using the notation we have just introduced as follows. Let ϕ be the formula $(x_1 + x_2) = x_3$. Then for all natural numbers i, j and k such that $i + j = k$:

- (a) $\vdash_Q \phi(i, j, k)$
- (b) $\vdash_Q \forall x_3 (\phi(i, j, x_3) \rightarrow x_3 = k)$

Put together (a) and (b) express the fact that for each pair of natural numbers i and j there is a unique value, k , for $i + j$. So they reflect the fact that addition is a function (with domain $\mathbb{N}^2$ and codomain $\mathbb{N}$), which gives a unique value for each pair of values for its two arguments. A consequence of (a) and (b) is that we can derive the sentence $(i + j) = k$ in Q only when $i + j = k$. We describe this situation by saying that *the formula ϕ represents the addition function in the theory Q* .

This conclusion does rely on the theory Q being consistent, which follows since Q has an interpretation, for instance $\mathcal{N}$. If Q had been inconsistent then, by Problem 3.2 of Unit 6, all sentences of the formal language would have been theorems of Q , including $(i + j) = n$ for all natural numbers n whether or not $i + j = n$.

Definition 2.1 Representability

Let T be a theory. A total function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ is said to be *representable* in T if there is a formula ϕ , in which no variables other than $x_1, x_2, \dots, x_n, x_{n+1}$ may occur freely, such that if $q_1, q_2, \dots, q_n, k$ are natural numbers with $f(q_1, q_2, \dots, q_n) = k$ then:

- (a) $\vdash_T \phi(q_1, q_2, \dots, q_n, k)$
- (b) $\vdash_T \forall x_{n+1} (\phi(q_1, q_2, \dots, q_n, x_{n+1}) \rightarrow x_{n+1} = k)$

In this case we say that the formula ϕ *represents* the function f in the theory T .

Example 2.4

As we have already remarked, it follows from Theorem 2.2 that the formula $(x_1 + x_2) = x_3$ represents the addition function in Q . Thus the addition function is representable in Q . ♦

Example 2.5

In a similar way it follows from Theorem 2.3 that the formula $(x_1 \cdot x_2) = x_3$ represents the multiplication function in Q . Thus the multiplication function is representable in Q . ♦

Example 2.6

Recall that the *zero function* is the function $\text{zero} : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $q \in \mathbb{N}$, $\text{zero}(q) = 0$. We show that this function is represented in Q by the formula $x_2 = 0$.

Take $q \in \mathbb{N}$. Then $\text{zero}(q) = 0$. Thus to show that the formula ϕ , namely $x_2 = 0$, represents the zero function in Q , we need to show that

$$\vdash_Q \phi(\mathbf{q}, 0) \quad \text{and} \quad \vdash_Q \forall x_2 (\phi(\mathbf{q}, x_2) \rightarrow x_2 = 0)$$

That is, we need to show that:

- (a) $\vdash_Q 0 = 0$
- (b) $\vdash_Q \forall x_2 (x_2 = 0 \rightarrow x_2 = 0)$

In fact, both of these formulas can be proved without the need to use any of the axioms of Q .

In case (a) we have a one-line proof, as follows.

$$(1) \quad 0 = 0 \quad \text{II}$$

The formal proof needed to show that (b) holds is also very short.

- | | | |
|---|---|-------|
| 1 | (1) $x_2 = 0$ | Ass |
| | (2) $(x_2 = 0 \rightarrow x_2 = 0)$ | CP, 1 |
| | (3) $\forall x_2 (x_2 = 0 \rightarrow x_2 = 0)$ | UI, 2 |

This completes the proof that the formula $x_2 = 0$ represents the zero function in Q . ◆

Problem 2.4

Recall that the *successor function* is the function $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $q \in \mathbb{N}$, $\text{succ}(q) = q + 1$. Show that this function is representable in Q .

Problem 2.5

Recall that if m and n are positive integers with $m \leq n$ then $U_m^n : \mathbb{N}^n \rightarrow \mathbb{N}$ is the *projection function* such that, for all $q_1, q_2, \dots, q_n \in \mathbb{N}$, $U_m^n(q_1, q_2, \dots, q_n) = q_m$. Show that this function is representable in Q .

You will recall that zero , succ and U_m^n for $0 < m \leq n$ are the basic primitive recursive functions which were introduced in *Unit 2*, Subsection 1.1. Thus the results of Example 2.6 and Problems 2.4 and 2.5 show that all the basic primitive recursive functions are representable in Q . What about more complicated total recursive functions? The next example gives us part of the answer to this question.

Example 2.7

Let $g_1 : \mathbb{N} \rightarrow \mathbb{N}$, $g_2 : \mathbb{N} \rightarrow \mathbb{N}$ and $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ be total functions which are representable in a theory T and let $h : \mathbb{N} \rightarrow \mathbb{N}$ be the total function defined by

$$h(q) = f(g_1(q), g_2(q)).$$

Then h is also representable in T .

Let ϕ_1 , ϕ_2 and ψ be formulas which represent the functions g_1 , g_2 and f respectively in T . Now consider the formula θ given by

$$\exists y_1 \exists y_2 ((\phi_1(x_1, y_1) \ \& \ \phi_2(x_1, y_2)) \ \& \ \psi(y_1, y_2, x_2))$$

where y_1 and y_2 are variables which do not occur in any of the formulas ϕ_1 , ϕ_2 and ψ . Observe that θ is a formula in which the variables x_1 and x_2 may occur freely and no other variables occur freely. We show that θ represents h in T .

Let q be some natural number. Suppose that $g_1(q) = q_1$, $g_2(q) = q_2$ and $f(q_1, q_2) = k$. Then

$$h(q) = f(g_1(q), g_2(q)) = f(q_1, q_2) = k.$$

Thus to show that θ represents h in T , we need to show that:

- (a) $\vdash_T \theta(\mathbf{q}, \mathbf{k})$
- (b) $\vdash_T \forall x (\theta(\mathbf{q}, x) \rightarrow x = \mathbf{k})$

Since $g_1(q) = q_1$ and ϕ_1 represents g_1 in T , we have

$$\vdash_T \phi_1(\mathbf{q}, \mathbf{q}_1) \quad (2.1)$$

Similarly, as $g_2(q) = q_2$ and ϕ_2 represents g_2 in T ,

$$\vdash_T \phi_2(\mathbf{q}, \mathbf{q}_2) \quad (2.2)$$

Likewise, as $f(q_1, q_2) = k$ and ψ represents f in T ,

$$\vdash_T \psi(\mathbf{q}_1, \mathbf{q}_2, \mathbf{k}) \quad (2.3)$$

Since the formula $((\phi_1(\mathbf{q}, \mathbf{q}_1) \& \phi_2(\mathbf{q}, \mathbf{q}_2)) \& \psi(\mathbf{q}_1, \mathbf{q}_2, \mathbf{k}))$ is a tautological consequence of the formulas that occur in (2.1), (2.2) and (2.3), we can deduce that

$$\vdash_T ((\phi_1(\mathbf{q}, \mathbf{q}_1) \& \phi_2(\mathbf{q}, \mathbf{q}_2)) \& \psi(\mathbf{q}_1, \mathbf{q}_2, \mathbf{k})) \quad (2.4)$$

We can apply the EI Rule twice to the formula in (2.4) and thus deduce that

$$\vdash_T \exists y_1 \exists y_2 ((\phi_1(\mathbf{q}, y_1) \& \phi_2(\mathbf{q}, y_2)) \& \psi(y_1, y_2, \mathbf{k}))$$

that is

$$\vdash_T \theta(\mathbf{q}, \mathbf{k})$$

This establishes that (a) holds.

Next, again using the fact that the formulas ϕ_1, ϕ_2 and ψ represent g_1, g_2 and f respectively in T , we have:

$$\vdash_T \forall x (\phi_1(\mathbf{q}, x) \rightarrow x = \mathbf{q}_1) \quad (2.5)$$

$$\vdash_T \forall x (\phi_2(\mathbf{q}, x) \rightarrow x = \mathbf{q}_2) \quad (2.6)$$

$$\vdash_T \forall x (\psi(\mathbf{q}_1, \mathbf{q}_2, x) \rightarrow x = \mathbf{k}) \quad (2.7)$$

The formula $\forall x (\theta(\mathbf{q}, x) \rightarrow x = \mathbf{k})$ can be derived using the formulas in (2.5), (2.6) and (2.7) as assumptions. Hence, it follows that

$$\vdash_T \forall x (\theta(\mathbf{q}, x) \rightarrow x = \mathbf{k})$$

thus establishing that (b) is also satisfied, and completing the proof that the formula θ represents the function h in the theory T . ♦

Problem 2.6

Show that

$$\chi_1, \chi_2, \chi_3 \vdash \forall x (\theta(\mathbf{q}, x) \rightarrow x = \mathbf{k})$$

where χ_1, χ_2 and χ_3 are, respectively, the formulas $\forall x (\phi_1(\mathbf{q}, x) \rightarrow x = \mathbf{q}_1)$, $\forall x (\phi_2(\mathbf{q}, x) \rightarrow x = \mathbf{q}_2)$ and $\forall x (\psi(\mathbf{q}_1, \mathbf{q}_2, x) \rightarrow x = \mathbf{k})$ and where θ is the formula $\exists y_1 \exists y_2 ((\phi_1(x_1, y_1) \& \phi_2(x_1, y_2)) \& \psi(y_1, y_2, x_2))$.

Note that in stating this result we have written ' $\vdash$ ' not ' $\vdash_Q$ ' or ' $\vdash_T$ ', because it is a result of quantifier logic which does not require any arithmetical axioms.

At the price only of additional notational complexity, the result of Example 2.7 can be generalized to cover the case where $h: \mathbb{N}^n \rightarrow \mathbb{N}$ is obtained by substitution from the functions $g_1: \mathbb{N}^n \rightarrow \mathbb{N}, g_2: \mathbb{N}^n \rightarrow \mathbb{N}, \dots, g_t: \mathbb{N}^n \rightarrow \mathbb{N}$ and $f: \mathbb{N}^t \rightarrow \mathbb{N}$ all of which are representable in a theory T .

Theorem 2.4

The set of total functions which are representable in a theory T is closed under substitution.

Recall that the recursive functions are those functions that can be obtained from the basic primitive recursive functions zero, succ and U_m^n using the operations of substitution, primitive recursion and minimization on a function a finite number of times. We have shown in Example 2.6 and Problems 2.4 and 2.5 that the basic primitive recursive functions are representable in Q , and Theorem 2.4 tells us that the set of total functions representable in Q is closed under substitution. In fact, all *total* recursive functions are representable in Q .

The proof of this is rather complicated. One difficulty is that the use of minimization does not always yield a total function, whereas our definition of representability applies only to total functions. One proof relies on an alternative characterization of the set of total recursive functions, obtained as follows. First we say that a function $f: \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ is *regular* if for each k -tuple $(n_1, n_2, \dots, n_k) \in \mathbb{N}^k$ there is some $y \in \mathbb{N}$ such that $f(n_1, n_2, \dots, n_k, y) = 0$. Then it can be shown that a function is total recursive if and only if it can be obtained from the functions add, mult, the characteristic function χ_{eq} of equality and the projection functions U_m^k by a finite number of applications of substitution and minimization of regular functions. The proof that all total recursive functions are representable in Q then requires showing, in addition to what we have already shown, that the functions add, mult and χ_{eq} are representable in Q and that the set of regular functions representable in Q is closed under minimization. We shall not give the proofs of these results, which we ask you to take on trust. The proofs make use of the axioms of Q , and so, in contrast to Theorem 2.4 which holds for all theories, these results and hence the result that all total recursive functions are representable in Q apply only to theories in which the axioms of Q are included as axioms or can be derived as theorems.

Before we can formally state the result that tells us that all total recursive functions are representable in Q , we need a definition.

Definition 2.2 Extension of a theory

Let S and T be two theories. We say that S *extends* T , or that S is an *extension* of T , if $T \subseteq S$. We often say that T is *weaker* than S and that S is *more powerful* than T .

By this definition, any theory T is an extension of itself.

It is straightforward to show that if S and T are theories such that, for each axiom ϕ of T , we have $\vdash_S \phi$, then S extends T .

We are now ready to state the promised result.

Theorem 2.5 Representability Theorem

Let T be a theory which extends Q . Then every total recursive function is representable in T .

Since Q counts as an extension of itself, this theorem does indeed tell us that every total recursive function is representable in Q , but it is worth stating this as a separate theorem.

Recursive functions were defined in Subsection 2.1 of Unit 3.

Note that this alternative characterization makes no mention of primitive recursion.

We invite you to show that χ_{eq} is representable in Q in Additional Exercise 2 for this section.

The Suggestions for Further Reading at the end of Unit 8 give references to places where these proofs may be found.

Theorem 2.6 Representability in Q

Every total recursive function is representable in Q .

Since Q is, as we have seen, quite a weak theory, it is perhaps a little surprising that it is powerful enough to represent all the total recursive functions.

It might also be thought that if T is a more powerful theory which extends Q , then more total functions might be representable in T . Surprisingly, this is not the case provided that T satisfies the following conditions.

- 1 T is consistent.
- 2 T has a set S of axioms for which there is an algorithm for deciding whether a given sentence is in S .

We sketch a proof of this claim. Suppose that T is an extension of Q which satisfies these conditions, and let $f: \mathbb{N}^n \rightarrow \mathbb{N}$ be a total function which is represented in T by the formula ϕ . Then the following both hold.

- (a) For all $q_1, q_2, \dots, q_n, k \in \mathbb{N}$,

$$\vdash_T \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k}) \iff f(q_1, q_2, \dots, q_n) = k.$$

- (b) There is an algorithm for systematically enumerating the theorems of T .

We ask you to verify (a) in Additional Exercise 3 for this section.

The truth of (b) can be seen as follows. We can systematically list all the finite sequences of formulas of our formal language and we can check which of these correspond to formal proofs of our system. For each formal proof we come across, we can decide on which assumptions the last line of the proof depends. Hence, if condition 2 holds, we can decide in which cases these assumptions are all axioms in S . Thus we can systematically list all those formulas which can be derived from assumptions which are axioms in S , that is, are theorems of T . So (b) holds.

We can now see that (a) and (b) imply that there is an algorithm for computing the values of the function f . Given $q_1, q_2, \dots, q_n \in \mathbb{N}$, we enumerate the theorems of T until we find one which has the form $\phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k})$. By (a) there will be precisely one theorem of this form. Once we have found this formula, we can extract from it the value of k and deduce that $f(q_1, q_2, \dots, q_n) = k$.

Thus f is algorithmically computable, and by appealing to Church's Thesis we can deduce that f is a recursive function.

We thus have the following theorem.

Church's Thesis was discussed in Unit 3. We appeal to Church's Thesis to simplify our informal argument. Theorem 2.7 can be proved without such an appeal.

Theorem 2.7

Suppose that T is a theory such that:

- (a) T is consistent;
- (b) T has a set S of axioms for which there is an algorithm for deciding whether a given sentence is in S .

Then every total function which is representable in T is recursive.

In Subsection 3.2 of *Unit 6*, we discussed criteria which we wanted axioms for number theory to satisfy. The first of these was that all the axioms should be true in the standard interpretation. Recall that it follows from this that all the theorems derived from the axioms are true in the standard interpretation. In consequence, any theory whose axioms are true in the standard interpretation is consistent, since there cannot be a sentence Φ such that both Φ and $\neg\Phi$ are true in the standard interpretation. The second criterion that we wanted the axioms of number theory to satisfy corresponds to the second condition of Theorem 2.7. Thus any theory whose axioms satisfy criteria 1 and 2 of Subsection 3.2 of *Unit 6* also satisfies the conditions of Theorem 2.7. In particular it follows that every total function which is representable in Q is recursive. Putting this together with Theorem 2.6 we have the following theorem.

This follows by Theorem 3.2 of *Unit 6*.

Theorem 2.8

A total function is representable in Q if and only if it is a recursive function.

This theorem provides a connection between our work on algorithms and recursive functions in *Units 1* to *3* and that on logic and formal proofs in *Units 4* to *7*. We chose to study the theory Q because it is the weakest theory for which the equivalence between *recursive* and *representable* total functions holds. It has the additional advantage that it has only a finite number of axioms.

Before we end this section, we give a technical result which will be useful in Section 3. For simplicity we state and prove this result in the case where f is a total function of one variable, and hence where ϕ is a formula which contains only two variables which may occur freely. First we need a preliminary result.

Theorem 2.9

Let ϕ be a formula in which x and y are the only variables which may occur freely. Then, for all natural numbers n and k ,

$$\phi(n, k), \forall y (\phi(n, y) \rightarrow y = k) \vdash \forall y (\phi(n, y) \leftrightarrow y = k)$$

As with Problem 2.6, we have written simply ' $\vdash$ ', rather than say ' $\vdash_Q$ ', as Theorem 2.9 is a result of quantifier logic which does not need any arithmetical axioms.

Proof

Consider the following schematic proof.

- | | | | |
|------|------|--|------------|
| 1 | (1) | $\phi(n, k)$ | Ass |
| 2 | (2) | $\forall y (\phi(n, y) \rightarrow y = k)$ | Ass |
| 2 | (3) | $(\phi(n, y) \rightarrow y = k)$ | UE, 2 |
| 4 | (4) | $y = k$ | Ass |
| | (5) | $y = y$ | II |
| 4 | (6) | $k = y$ | Sub, 4, 5 |
| 1, 4 | (7) | $\phi(n, y)$ | Sub, 1, 6 |
| 1 | (8) | $(y = k \rightarrow \phi(n, y))$ | CP, 7 |
| 1, 2 | (9) | $(\phi(n, y) \leftrightarrow y = k)$ | Taut, 3, 8 |
| 1, 2 | (10) | $\forall y (\phi(n, y) \leftrightarrow y = k)$ | UI, 9 |

This establishes the desired conclusion. ■

Theorem 2.10

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a total function which is represented in the theory T by the formula ϕ . Then, for all $n, k \in \mathbb{N}$, if $f(n) = k$ then

$$\vdash_T \forall y (\phi(\mathbf{n}, y) \leftrightarrow y = \mathbf{k})$$

Proof

Suppose that the formula ϕ represents the function $f: \mathbb{N} \rightarrow \mathbb{N}$ in the theory T . Let n and k be natural numbers such that $f(n) = k$. Then, by Definition 2.1, we have:

- (a) $\vdash_T \phi(\mathbf{n}, \mathbf{k})$
- (b) $\vdash_T \forall y (\phi(\mathbf{n}, y) \rightarrow y = \mathbf{k})$

It then follows from (a) and (b), using Theorem 2.9, that

$$\vdash_T \forall y (\phi(\mathbf{n}, y) \leftrightarrow y = \mathbf{k})$$

Theorem 2.10 can easily be extended to total functions of more than one variable.

We end this section with a remark about Church's Thesis. One of the reasons we gave, in *Unit 3*, for believing Church's Thesis was the stability of the definition of recursive functions. We have now seen that for total functions the definition is equivalent to 'representable in the theory Q '. Theorems 2.7 and 2.8 thus give us another example of the stability of the definition. They show that, if T is an extension of Q which satisfies the conditions of Theorem 2.7, then the set of total functions which are representable in T is precisely the set of total recursive functions and hence is no greater than the set of total functions representable in Q itself.

Note that, although our informal argument leading to Theorem 2.7 used Church's Thesis, it could have been avoided, so that there is no circularity here.

3 GÖDEL'S DIAGONAL LEMMA

In the standard interpretation, the formulas of our formal language are interpreted as expressing properties of natural numbers. In *Unit 4*, Subsection 2.2, we associated with each formula ϕ of the language a natural number denoted by $\gamma(\phi)$ and called the Gödel number of ϕ . Thus a formula which expresses a property of natural numbers can be interpreted as expressing a property of formulas. The following example should help make this clear.

Let ϕ be the following formula.

$$(\exists y x = (2 \cdot y) \ \& \ \neg \exists y x = (4 \cdot y))$$

In the standard interpretation $\mathcal{N}$, ϕ expresses the property that x is divisible by 2 but not by 4. Such a number x has the form $2^1 3^{n_2} \dots p_k^{n_k}$ where $n_2, \dots, n_k$ are natural numbers (which could be 0) and p_j is the j th prime number. The Gödel number of a formula has this form only when the first symbol of the formula is the left-hand bracket. Thus we can interpret ϕ as expressing the following property of formulas: 'if x is the Gödel number of a formula, then it is the Gödel number of a formula which begins with the symbol '('.

In this way, a formula ϕ can be interpreted as referring to other formulas. So the possibility exists of constructing a formula which refers to itself. Self-reference of this kind is a key idea in the proofs of the theorems in *Unit 8*. We shall explore self-reference in this section, culminating in *Gödel's Diagonal Lemma*, which will prove of importance in *Unit 8*. We begin by discussing the concept of *diagonalization*.

Recall that x is the Gödel number of a string of symbols if and only if it has the form $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ where p_j is the j th prime number and $n_1, n_2, \dots, n_k$ are all positive integers.

Of course the formula ϕ also expresses a property of numbers which do not code strings of symbols, but here we are only interested in the property ϕ expresses of formulas, or more precisely of the formulas as given by Gödel numbers.

3.1 Diagonalization of a formula

In this subsection we take the first step in the construction of a self-referring formula. We introduce the notion of the *diagonalization* of a formula.

Recall from Subsection 1.1 that, for each natural number n , the symbol $\mathbf{n}$ is an abbreviation for the term which consists of the symbol $\mathbf{0}$ followed by n occurrences of $'$. Now let ϕ be some formula whose Gödel number is n . Then we define $\ulcorner \phi \urcorner$ to be the term $\mathbf{n}$. Note carefully the distinction: $\gamma(\phi)$ is the natural number which is the Gödel number of ϕ , while $\ulcorner \phi \urcorner$ is the term of our formal language which in the standard interpretation is taken as referring to the number $\gamma(\phi)$. For example, if ϕ is the formula $\exists x_1 x_0 = (\mathbf{0}'' \cdot x_1)$, then $\gamma(\phi)$ is the number

$$2^9 3^{16} 5^{15} 7^{14} 11^{11} 13^{10} 17^{11} 19^{11} 23^{13} 29^{16} 31^2$$

and $\ulcorner \phi \urcorner$ is the term which consists of the symbol $\mathbf{0}$ followed by $\gamma(\phi)$ occurrences of the symbol $'$. In what follows $\ulcorner \phi \urcorner$ will be seen to behave like a name for the formula ϕ .

Definition 3.1 Diagonalization

Let ϕ be a formula in which only the variable z may occur freely. The *diagonalization* of ϕ is the sentence

$$\exists z (z = \ulcorner \phi \urcorner \ \& \ \phi)$$

which we denote by D_ϕ .

In this definition, z could be *any* variable x_i . We would need to know specifically which variable it stood for before we could calculate the Gödel number $\gamma(\phi)$ of ϕ and hence $\ulcorner \phi \urcorner$.

Notice that if ϕ is a formula in which only the variable z may occur freely, then D_ϕ is the sentence which, in the standard interpretation, expresses the fact that the number $\gamma(\phi)$ satisfies ϕ .

Example 3.1

Let ϕ be the formula

$$(\exists y x = (\mathbf{2} \cdot y) \ \& \ \neg \exists y x = (\mathbf{4} \cdot y))$$

in which only x occurs freely. $\gamma(\phi)$ is a very large number which we shall not write out in full. However, we note that the prime decomposition of $\gamma(\phi)$ begins $2^{13} 3^9 \dots$. We shall let m be this number.

Then D_ϕ , the diagonalization of ϕ , is the sentence

$$\exists x (x = \mathbf{m} \ \& \ (\exists y x = (\mathbf{2} \cdot y) \ \& \ \neg \exists y x = (\mathbf{4} \cdot y)))$$

Since the number m is divisible by 2, but is not divisible by 4, we see that D_ϕ is true in the standard interpretation.

However, in general the diagonalization of a formula need not be true in the standard interpretation. For example, let ψ be the formula $\neg \exists y x = (\mathbf{2} \cdot y)$, and let n be $\gamma(\psi)$, that is, the Gödel number of ψ . Then D_ψ is the sentence

$$\exists x (x = \mathbf{n} \ \& \ \neg \exists y x = (\mathbf{2} \cdot y))$$

Now, n is the Gödel number of a formula and hence is divisible by 2. Thus D_ψ is false under the standard interpretation. ♦

Problem 3.1

Let ϕ be the formula of Example 3.1. Write down the sentence $D_{\neg \phi}$, that is, the diagonalization of the formula $\neg \phi$. Is $D_{\neg \phi}$ true in the standard interpretation?

It might seem more natural to define the diagonalization of ϕ to be $\phi(\ulcorner \phi \urcorner)$, that is, the sentence we obtain from ϕ by replacing all the free occurrences of z by the term $\ulcorner \phi \urcorner$. There are technical reasons why this turns out not to be a good choice. However, it is important that we can prove in our formal system that D_ϕ is equivalent to $\phi(\ulcorner \phi \urcorner)$, as we now demonstrate.

Theorem 3.1

For each formula ϕ in which only the variable z may occur freely and for each natural number n ,

$$\vdash (\exists z (z = n \ \& \ \phi) \leftrightarrow \phi(n))$$

If you want more practice in proving formulas in quantifier logic, try proving this theorem yourself before reading our proof.

Proof

We give the following schematic formal proof.

1	(1)	$\exists z (z = n \ \& \ \phi)$	Ass
2	(2)	$(z = n \ \& \ \phi)$	Ass
2	(3)	$z = n$	Taut, 2
2	(4)	ϕ	Taut, 2
2	(5)	$\phi(n)$	Sub, 3, 4
1	(6)	$\phi(n)$	EH, 5
	(7)	$(\exists z (z = n \ \& \ \phi) \rightarrow \phi(n))$	CP, 6
8	(8)	$\phi(n)$	Ass
	(9)	$n = n$	Π
8	(10)	$(n = n \ \& \ \phi(n))$	Taut, 8, 9
8	(11)	$\exists z (z = n \ \& \ \phi)$	EI, 10
	(12)	$(\phi(n) \rightarrow \exists z (z = n \ \& \ \phi))$	CP, 11
	(13)	$(\exists z (z = n \ \& \ \phi) \leftrightarrow \phi(n))$	Taut, 7, 12

Since there are no assumptions in force on line 13, we can deduce that $\vdash (\exists z (z = n \ \& \ \phi) \leftrightarrow \phi(n))$. ■

Theorem 3.2

For each formula ϕ in which only the variable z may occur freely,

$$\vdash (D_\phi \leftrightarrow \phi(\ulcorner \phi \urcorner))$$

Notice that Theorems 3.1 and 3.2 are results about quantifier logic, hence the use of ' $\vdash$ '.

Proof

Let n be the Gödel number of the formula ϕ . Then D_ϕ is the sentence $\exists z (z = n \ \& \ \phi)$ and $\phi(\ulcorner \phi \urcorner)$ is identical to $\phi(n)$. Thus this theorem is an immediate consequence of Theorem 3.1. ■

3.2 Gödel's idea

In this subsection we shall prove Gödel's Diagonal Lemma, a key result on self-reference of formulas. We need first to show that there is a total recursive function which gives the Gödel number of the diagonalization, D_ϕ , of a formula ϕ , in terms of the Gödel number of ϕ . We begin by reminding you briefly of the results from earlier units which we need in order to do this.

First recall from Subsection 2.2 of *Unit 4* that we assign a positive integer $\beta(\sigma_i)$ to each basic symbol σ_i of our formal language, and that if σ is the string of symbols $\sigma_1\sigma_2\ldots\sigma_k$ we assign to σ the Gödel number $\gamma(\sigma)$ given by

$$\gamma(\sigma) = p_1^{\beta(\sigma_1)} p_2^{\beta(\sigma_2)} \ldots p_k^{\beta(\sigma_k)}$$

where p_j is the j th prime number.

Next we need to recall that in Section 3 of *Unit 2* we saw that each of the following three functions is primitive recursive.

- (a) $n \mapsto p_n$, where $p_0 = 1$ and, for $n \geq 1$, p_n is the n th prime number.
- (b) len , where $\text{len}(n) = \begin{cases} \text{the number of distinct} \\ \text{prime divisors of } n, & \text{if } n \geq 2, \\ 0, & \text{otherwise.} \end{cases}$
- (c) $(n, j) \mapsto (n)_j$, where $(n)_j$ is the exponent of the j th prime number p_j in the prime decomposition of n for $n \geq 2$ and is 0 otherwise.

If n codes the string of symbols $\sigma_1\sigma_2\ldots\sigma_k$, so that $n = \gamma(\sigma_1\sigma_2\ldots\sigma_k)$, then, as $\beta(\sigma_i) > 0$ for any symbol σ_i , the number of symbols k is given by $\text{len}(n)$. And, for $1 \leq i \leq k = \text{len}(n)$, we have $(n)_i = \beta(\sigma_i)$.

Our first step towards proving that $\gamma(D_\phi)$ is given by a recursive function of $\gamma(\phi)$ is to prove the following theorem. In it, it will be convenient to use the notation $\sigma\tau$ for the concatenation of two strings σ and τ : that is, if σ is the string $\sigma_1\sigma_2\ldots\sigma_k$ and τ is the string $\tau_1\tau_2\ldots\tau_l$, then we write $\sigma\tau$ for the string $\sigma_1\sigma_2\ldots\sigma_k\tau_1\tau_2\ldots\tau_l$. For example, if σ is $\forall x$ and τ is $(x + 0) = x$, then $\sigma\tau$ is $\forall x (x + 0) = x$.

Theorem 3.3

There is a primitive recursive function $*$ such that, for all strings σ and τ of basic symbols of our formal language, if $m = \gamma(\sigma)$ and $n = \gamma(\tau)$ then $m * n = \gamma(\sigma\tau)$.

Proof

Suppose that σ is the string $\sigma_1\sigma_2\ldots\sigma_k$ and τ is the string $\tau_1\tau_2\ldots\tau_l$. Let $m = \gamma(\sigma)$ and $n = \gamma(\tau)$, so that $m = p_1^{\beta(\sigma_1)} p_2^{\beta(\sigma_2)} \ldots p_k^{\beta(\sigma_k)}$ and $n = p_1^{\beta(\tau_1)} p_2^{\beta(\tau_2)} \ldots p_l^{\beta(\tau_l)}$. Note that $k = \text{len}(m)$, $l = \text{len}(n)$ and, for $1 \leq i \leq l$, $(n)_i = \beta(\tau_i)$. Then

$$\begin{aligned} m * n &= \gamma(\sigma\tau) = p_1^{\beta(\sigma_1)} p_2^{\beta(\sigma_2)} \ldots p_k^{\beta(\sigma_k)} p_{k+1}^{\beta(\tau_1)} p_{k+2}^{\beta(\tau_2)} \ldots p_{k+l}^{\beta(\tau_l)} \\ &= m \times \prod_{i=1}^l p_{k+i}^{\beta(\tau_i)} \\ &= m \times \prod_{i=1}^{\text{len}(n)} p_{\text{len}(m)+i}^{(n)_i}. \end{aligned}$$

Note that the formula for $m * n$ provides a value for $m * n$ even when either m and/or n does not code a string of basic symbols, but we are not interested in the value of $m * n$ in such cases.

We have already noted that the functions len , $n \mapsto p_n$ and $(n, j) \mapsto (n)_j$ are primitive recursive. We know from *Unit 2* that multiplication, exponentiation and addition are primitive recursive. We also saw in Theorem 3.2 of *Unit 2* that primitive recursive functions are closed under bounded products. It therefore follows that the function given by the above formula is primitive recursive. ■

Before we go on to use Theorem 3.3 to help prove that $\gamma(D_\phi)$ is given by a recursive function of $\gamma(\phi)$, we give some related results, two of which will also prove useful in deriving our result about $\gamma(D_\phi)$.

Example 3.2

There is a primitive recursive function, neg , such that if n is the Gödel number of a formula ϕ then $\text{neg}(n)$ is the Gödel number of its negation $\neg\phi$.

Recall that $\beta(\neg) = 7$, and hence the string $\neg$ consisting of the negation symbol by itself has Gödel number 2^7 , that is $\gamma(\neg) = 2^7$. Therefore, if $\gamma(\phi) = n$, $\gamma(\neg\phi) = 2^7 * n$. Thus we can put

$$\text{neg}(n) = 2^7 * n.$$

Since $*$ is primitive recursive, it follows that the function neg is primitive recursive. $\blacklozenge$

Problem 3.2

Show that there is a primitive recursive function, conj , such that if m is the Gödel number of a formula ϕ and n is the Gödel number of a formula ψ then $\text{conj}(m, n)$ is the Gödel number of their conjunction $(\phi \ \& \ \psi)$.

Problem 3.3

Show that there is a primitive recursive function, num , such that, for each natural number n , $\text{num}(n)$ is the Gödel number of the term $\mathbf{n}$.

We can now prove that the Gödel number of D_ϕ is given by a primitive recursive function.

Theorem 3.4

There is a primitive recursive function, diag , such that if n is the Gödel number of a formula ϕ in which only the variable z may occur freely then $\text{diag}(n)$ is the Gödel number of its diagonalization D_ϕ .

Proof

Let ϕ be a formula and suppose that n is the Gödel number of ϕ . Then D_ϕ , the diagonalization of ϕ , is the formula

$$\exists z (z = \mathbf{n} \ \& \ \phi)$$

Suppose that z is the variable x_i . Then the string ' $z =$ ' has Gödel number $2^{15+i}3^{14}$, and hence the formula $z = \mathbf{n}$ has Gödel number $2^{15+i}3^{14} * \text{num}(n)$.

Therefore the conjunction $(z = \mathbf{n} \ \& \ \phi)$ has Gödel number

$\text{conj}(2^{15+i}3^{14} * \text{num}(n), n)$. The string $\exists z$ has Gödel number 2^93^{15+i} and hence $\exists z (z = \mathbf{n} \ \& \ \phi)$ has Gödel number $2^93^{15+i} * \text{conj}(2^{15+i}3^{14} * \text{num}(n), n)$.

Thus we can put

$$\text{diag}(n) = 2^93^{15+i} * \text{conj}(2^{15+i}3^{14} * \text{num}(n), n).$$

Since, by Theorem 3.3, the function $*$ is primitive recursive and, by Problems 3.2 and 3.3, conj and num are primitive recursive, it follows that diag is primitive recursive. $\blacksquare$

Recall that our aim in this section is to prove Gödel's Diagonal Lemma, a key result on self-reference of formulas. We are now in a position to move rapidly towards doing this.

First, suppose that χ is a formula in which z is the only free variable. Then, as we saw at the start of this section, the formula χ can be interpreted as saying that, if z is the Gödel number of a formula, then it is the Gödel number of a formula which has a certain property. In other words, provided z is the Gödel number of a formula, χ expresses the fact that ‘the formula with Gödel number z has the property expressed by χ ’. Next consider what happens if χ is the formula $\exists y (y = z \ \& \ \psi)$, where ψ is some formula in which only the variable y may occur freely. Putting $z = \ulcorner \psi \urcorner$ gives the formula D_ψ , the diagonalization of ψ , which can thus be interpreted as saying that the formula ψ has the given property. However, we have not yet achieved self-reference, because D_ψ expresses a fact about ψ , not about D_ψ . To achieve self-reference, Gödel invented a clever device for constructing a sentence G_ψ which is logically equivalent to the sentence $\psi(\ulcorner G_\psi \urcorner)$. The sentence $\psi(\ulcorner G_\psi \urcorner)$ expresses the fact that ‘the sentence G_ψ has the property expressed by ψ ’. Since G_ψ is logically equivalent to $\psi(\ulcorner G_\psi \urcorner)$ it can be regarded as expressing the same thing. So G_ψ can be regarded as expressing the fact that ‘the sentence G_ψ has the property expressed by ψ ’ and in this way we have achieved self-reference: the sentence G_ψ refers to itself.

Recall that $\psi(\ulcorner G_\psi \urcorner)$ denotes the formula ψ with all free occurrences of its only free variable, y , replaced by $\ulcorner G_\psi \urcorner$.

So, to achieve this self-reference, we need to construct a sentence G_ψ that is logically equivalent to $\psi(\ulcorner G_\psi \urcorner)$. This is Gödel’s Diagonal Lemma. The full proof of this key result involves some technicalities which can obscure the main idea. So we begin with a sketch of Gödel’s construction before giving a precise proof.

We start with a theory T in which the function diag of Theorem 3.4 can be represented. Since diag is primitive recursive, it is also total, so that, by the Representability Theorem, T could be the theory Q or any extension of it.

Let ψ be a formula in which y is the only variable which may occur freely. Now we let θ be the ‘formula’

$$\exists y (y = \text{diag}(x) \ \& \ \psi)$$

We have put ‘formula’ in inverted commas because, of course, θ is strictly not a formula of our formal language. This is because ‘ $y = \text{diag}(x)$ ’ is not a formula of this language. However, as the function diag is representable in T , when it comes to the precise proof we are able to replace ‘ $y = \text{diag}(x)$ ’ by a formula of our formal language. Note that x is the only variable which occurs freely in θ .

We now consider the formula D_θ , the diagonalization of θ . By Theorem 3.2, $\vdash_T (D_\theta \leftrightarrow \theta(\ulcorner \theta \urcorner))$. Hence, by the definition of θ ,

$$\vdash_T (D_\theta \leftrightarrow \exists y (y = \text{diag}(\ulcorner \theta \urcorner) \ \& \ \psi)) \quad (3.1)$$

Now, by definition, $\text{diag}(\ulcorner \theta \urcorner)$ is the Gödel number of D_θ and hence ‘ $y = \text{diag}(\ulcorner \theta \urcorner)$ ’ is equivalent to $y = \ulcorner D_\theta \urcorner$, so that we can rewrite (3.1) as

See Theorem 3.4.

$$\vdash_T (D_\theta \leftrightarrow \exists y (y = \ulcorner D_\theta \urcorner \ \& \ \psi)) \quad (3.2)$$

Now, by Theorem 3.1,

$$\vdash_T (\exists y (y = \ulcorner D_\theta \urcorner \ \& \ \psi) \leftrightarrow \psi(\ulcorner D_\theta \urcorner)) \quad (3.3)$$

Hence, from (3.2) and (3.3), we have

$$\vdash_T (D_\theta \leftrightarrow \psi(\ulcorner D_\theta \urcorner))$$

which tells us that D_θ is logically equivalent to $\psi(\ulcorner D_\theta \urcorner)$. Thus, if we take G_ψ to be the sentence D_θ , we have obtained the desired sentence G_ψ which is logically equivalent to $\psi(\ulcorner G_\psi \urcorner)$.

We now give the precise proof.

Theorem 3.5 Gödel's Diagonal Lemma

Let T be a theory in which the function diag is representable. Then, for each formula ψ in which y is the only variable which may occur freely, there is a sentence G_ψ such that

$$\vdash_T (G_\psi \leftrightarrow \psi(\ulcorner G_\psi \urcorner))$$

Proof

Let ϕ be a formula which represents the function diag in the theory T . Since diag is a function from $\mathbb{N}$ to $\mathbb{N}$, the definition of representability tells us that only two variables, x and y say, can occur freely in ϕ .

Now let ψ be a formula in which y is the only variable which may have free occurrences, and let θ be the formula

$$\exists y (\phi \& \psi)$$

Note that x is the only variable which may occur freely in θ .

We shall show that we can take G_ψ to be the formula D_θ , that is, the diagonalization of θ .

Let n be the Gödel number of θ . By Theorem 3.2, $\vdash_T (D_\theta \leftrightarrow \theta(\mathbf{n}))$, as $\mathbf{n} = \ulcorner \theta \urcorner$. Hence, using the definition of θ , we have

$$\vdash_T (D_\theta \leftrightarrow \exists y (\phi(\mathbf{n}, y) \& \psi)) \quad (3.4) \quad \text{Recall that } \phi(\mathbf{n}, y) \text{ denotes the formula } \phi \text{ with all the free occurrences of } x \text{ replaced by } \mathbf{n}.$$

Now let k be the Gödel number of D_θ . So $k = \text{diag}(n)$, and hence, as ϕ represents diag in T , it follows from Theorem 2.10 that

$$\vdash_T \forall y (\phi(\mathbf{n}, y) \leftrightarrow y = k) \quad (3.5)$$

We now make use of the result in Additional Exercise 5 for Section 1 of Unit 6. If in this result we let δ be the formula D_θ , χ be the formula $\phi(\mathbf{n}, y)$ and γ be the formula $y = k$, we see that from (3.4) and (3.5) it follows that

$$\vdash_T (D_\theta \leftrightarrow \exists y (y = k \& \psi)) \quad (3.6)$$

Now, by Theorem 3.1,

$$\vdash_T (\exists y (y = k \& \psi) \leftrightarrow \psi(\mathbf{k})) \quad (3.7)$$

The Tautology Rule applied to (3.6) and (3.7) gives

$$\vdash_T (D_\theta \leftrightarrow \psi(\mathbf{k})) \quad (3.8)$$

Since k is the Gödel number of D_θ , $\mathbf{k}$ is identical to $\ulcorner D_\theta \urcorner$. So we can deduce from (3.8) that

$$\vdash_T (D_\theta \leftrightarrow \psi(\ulcorner D_\theta \urcorner))$$

Thus, defining G_ψ to be the formula D_θ , we have that

$$\vdash_T (G_\psi \leftrightarrow \psi(\ulcorner G_\psi \urcorner))$$

and hence the proof of the theorem is completed. ■

This result says that, for all formulas γ , δ , ϕ and ψ ,

$$(\delta \leftrightarrow \exists y (\chi \& \psi), \forall y (\chi \leftrightarrow \gamma) \vdash (\delta \leftrightarrow \exists y (\gamma \& \psi)))$$

SUMMARY

We began by exploring examples of theorems of the theory Q . We discovered that, although Q is a very weak theory, it is powerful enough for us to be able to prove equations and inequations involving numerical terms. This led us to the idea of a formula *representing* a total function in a given theory. Although we omitted many of the technical details of the proof, we indicated that the total functions representable in the theory Q are precisely the total recursive functions. In the final section we introduced the idea of the *diagonalization* of a formula. This enabled us to prove Gödel's Diagonal Lemma which shows how, given a formula expressing properties of numbers, we can achieve self-reference, an idea which will play a key role in Unit 8.

OBJECTIVES

We list those topics on which we may set assessment questions to test your understanding of this unit.

After working through this unit you should be able to:

- construct simple formal proofs in the theory Q ;
- understand what it means for a total function to be representable in a theory T ;
- in the case of simple functions, show that they are representable in the theory Q ;
- understand the content and significance of the Representability Theorem;
- write down and interpret the diagonalization of a given formula;
- explain the meaning of Gödel's Diagonal Lemma.

ADDITIONAL EXERCISES

Most of these exercises provide further practice, should you feel you need it, in handling the main ideas in the unit on which you are likely to be assessed. There are no additional exercises on Section 3.

There are a few harder problems, labelled as such in the margin. These are harder than any of the problems you are likely to encounter in the assessment and are included solely as challenges for the interested student.

Section 1

- 1 Show the following.
 - (a) $\vdash_Q \exists x \forall y (y' + x) = y'$
 - (b) $\vdash_Q \forall x (\mathbf{0} + (x \cdot \mathbf{0})) = ((\mathbf{0} + x) \cdot \mathbf{0})$
 - (c) $\vdash_Q \exists x \neg (x + x) = (x + x)''$
 - (d) $\vdash_Q \forall x \exists y (x + y) = x''$
- 2 One of the following sentences is a theorem of Q and one is not. Give a formal proof of the sentence that is a theorem of Q and explain why the other sentence is not a theorem of Q .
 - (a) $\forall x (x \cdot (\mathbf{0}' + \mathbf{0})) = ((x \cdot \mathbf{0}') + (x \cdot \mathbf{0}))$
 - (b) $\forall x ((\mathbf{0}' + \mathbf{0}) \cdot x) = ((\mathbf{0}' \cdot x) + (\mathbf{0} \cdot x))$

- 3 Prove that, if u and v are any two distinct variables,

$$\vdash_Q \forall u \forall v (u + v') = (u + v)'$$

Hint: The sentence you are asked to derive is axiom $Q5$ with the variable x replaced by u and the variable y replaced by v . You should therefore find Example 1.8 helpful.

Section 2

- 1 Show that, for all natural numbers i ,

$$\vdash_Q \forall x (x' + \mathbf{i}) = (x + \mathbf{i})'$$

Harder problem

Hint: Prove using Mathematical Induction that, for all natural numbers i , $P(i)$ holds, where $P(i)$ is the proposition $\vdash_Q \forall x (x' + \mathbf{i}) = (x + \mathbf{i})'$. See Theorem 2.2 for an example of this type of proof.

- 2 Show that χ_{eq} , the characteristic function of the equality relation, is representable in Q . Recall that χ_{eq} is defined by

Harder problem

$$\chi_{eq}(q_1, q_2) = \begin{cases} 1, & \text{if } q_1 = q_2, \\ 0, & \text{if } q_1 \neq q_2. \end{cases}$$

- 3 Let T be a theory which extends Q and suppose that the formula ϕ represents the total function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ in the theory T .
 - (a) Show that for, all $q_1, q_2, \dots, q_n, j \in \mathbb{N}$, if $f(q_1, q_2, \dots, q_n) \neq j$ then

Harder problem

$$\vdash_T \neg \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{j})$$

- (b) Deduce that, if T is consistent, then, for all $q_1, q_2, \dots, q_n, k \in \mathbb{N}$,

$$\vdash_T \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k}) \iff f(q_1, q_2, \dots, q_n) = k$$

SOLUTIONS TO THE PROBLEMS

Solution 1.1

Consider the interpretation U whose domain is the set with one element u , which interprets 0 , and for which $u = u'$, so that $u' = u''$. Thus the sentence $1 \neq 2$ is not true in this interpretation. If it were the case that $\vdash 1 \neq 2$ then, by the Correctness Theorem, this sentence would be true in every interpretation of the formal language. Thus it is not the case that $\vdash 1 \neq 2$.

This interpretation was used in the solution to Additional Exercise 4 for Section 3 of Unit 4.

Solution 1.2

- | | | | | |
|-----|---------|------|--|-------------|
| (a) | 1 | (1) | $\forall x \forall y (x' = y' \rightarrow x = y)$ | Ass |
| | 2 | (2) | $\forall x \neg 0 = x'$ | Ass |
| | 3 | (3) | $0'' = 0'''$ | Ass |
| | 1 | (4) | $\forall y (0'' = y' \rightarrow 0' = y)$ | UE, 1 |
| | 1 | (5) | $(0'' = 0''' \rightarrow 0' = 0'')$ | UE, 4 |
| | 1, 3 | (6) | $0' = 0''$ | Taut, 3, 5 |
| | 1 | (7) | $\forall y (0' = y' \rightarrow 0 = y)$ | UE, 1 |
| | 1 | (8) | $(0' = 0'' \rightarrow 0 = 0')$ | UE, 7 |
| | 1, 3 | (9) | $0 = 0'$ | Taut, 6, 8 |
| | 2 | (10) | $\neg 0 = 0'$ | UE, 2 |
| | 1, 2, 3 | (11) | $(0 = 0' \& \neg 0 = 0')$ | Taut, 9, 10 |
| | 1, 2 | (12) | $(0'' = 0''' \rightarrow (0 = 0' \& \neg 0 = 0'))$ | CP, 11 |
| | 1, 2 | (13) | $\neg 0'' = 0'''$ | Taut, 12 |

As assumptions 1 and 2 are axioms $Q1$ and $Q2$, we have shown that $\vdash_Q \neg 0'' = 0'''$.

- | | | | | |
|-----|---|-----|-------------------------|-------|
| (b) | 1 | (1) | $\forall x \neg 0 = x'$ | Ass |
| | 1 | (2) | $\neg 0 = 0'''$ | UE, 1 |

As assumption 1 is axiom $Q2$, we have shown that $\vdash_Q \neg 0 = 0'''$.

- | | | | | |
|-----|---------|------|--|------------|
| (c) | 1 | (1) | $\forall x \forall y (x' = y' \rightarrow x = y)$ | Ass |
| | 2 | (2) | $\forall x \neg 0 = x'$ | Ass |
| | 3 | (3) | $0'''' = 0'$ | Ass |
| | 1 | (4) | $\forall y (0'''' = y' \rightarrow 0''' = y)$ | UE, 1 |
| | 1 | (5) | $(0'''' = 0' \rightarrow 0''' = 0)$ | UE, 4 |
| | 1, 3 | (6) | $0''' = 0$ | Taut, 3, 5 |
| | 2 | (7) | $\neg 0 = 0'''$ | UE, 2 |
| | | (8) | $0''' = 0'''$ | II |
| | 1, 3 | (9) | $0 = 0'''$ | Sub, 6, 8 |
| | 1, 2, 3 | (10) | $(0 = 0''' \& \neg 0 = 0''')$ | Taut, 7, 9 |
| | 1, 2 | (11) | $(0'''' = 0' \rightarrow (0 = 0''' \& \neg 0 = 0'''))$ | CP, 10 |
| | 1, 2 | (12) | $\neg 0'''' = 0'$ | Taut, 11 |

As assumptions 1 and 2 are axioms $Q1$ and $Q2$, we have shown that $\vdash_Q \neg 0'''' = 0'$.

Note that the formula on line 7 is not the negation of the formula on line 6, but that the use of the II and Sub Rules on lines 8 and 9 give us a suitable formula with which to obtain a contradiction on line 10.

Solution 1.3

- | | | | |
|------|-----|---|-----------|
| 1 | (1) | $\forall x (x + 0) = x$ | Ass |
| 2 | (2) | $\forall x \forall y (x + y') = (x + y)'$ | Ass |
| 2 | (3) | $\forall y (0 + y') = (0 + y)'$ | UE, 2 |
| 2 | (4) | $(0 + 0') = (0 + 0)'$ | UE, 3 |
| 1 | (5) | $(0 + 0) = 0$ | UE, 1 |
| 1, 2 | (6) | $(0 + 0') = 0'$ | Sub, 4, 5 |

As assumptions 1 and 2 are axioms Q4 and Q5, we have shown that $\vdash_Q (0 + 0') = 0'$.

Solution 1.4

- | | | | |
|------------|------|--|-------------|
| 1 | (1) | $\forall x (x + 0) = x$ | Ass |
| 2 | (2) | $\forall x \forall y (x + y') = (x + y)'$ | Ass |
| 3 | (3) | $\forall x (x \cdot 0) = 0$ | Ass |
| 4 | (4) | $\forall x \forall y (x \cdot y') = ((x \cdot y) + x)$ | Ass |
| 4 | (5) | $\forall y (0' \cdot y') = ((0' \cdot y) + 0')$ | UE, 4 |
| 4 | (6) | $(0' \cdot 0') = ((0' \cdot 0) + 0')$ | UE, 5 |
| 3 | (7) | $(0' \cdot 0) = 0$ | UE, 3 |
| 3, 4 | (8) | $(0' \cdot 0') = (0 + 0')$ | Sub, 6, 7 |
| 2 | (9) | $\forall y (0 + y') = (0 + y)'$ | UE, 2 |
| 2 | (10) | $(0 + 0') = (0 + 0)'$ | UE, 9 |
| 1 | (11) | $(0 + 0) = 0$ | UE, 1 |
| 1, 2 | (12) | $(0 + 0') = 0'$ | Sub, 10, 11 |
| 1, 2, 3, 4 | (13) | $(0' \cdot 0') = 0'$ | Sub, 8, 12 |

As assumptions 1, 2, 3 and 4 are axioms Q4, Q5, Q6 and Q7, we have shown that $\vdash_Q (0' \cdot 0') = 0'$.

Solution 1.5

We use proof by contradiction. Our proof assumes $(1 + 1) = 1$. It essentially first derives $(1 + 1) = 2$, to obtain $1 = 2$, and then derives a contradiction from this.

- | | | | |
|----------------|------|--|--------------|
| 1 | (1) | $\forall x (x + 0) = x$ | Ass |
| 2 | (2) | $\forall x \forall y (x + y') = (x + y)'$ | Ass |
| 3 | (3) | $(0' + 0') = 0'$ | Ass |
| 2 | (4) | $\forall y (0' + y') = (0' + y)'$ | UE, 2 |
| 2 | (5) | $(0' + 0') = (0' + 0)'$ | UE, 4 |
| 1 | (6) | $(0' + 0) = 0'$ | UE, 1 |
| 1, 2 | (7) | $(0' + 0') = 0''$ | Sub, 5, 6 |
| 1, 2, 3 | (8) | $0' = 0''$ | Sub, 3, 7 |
| 9 | (9) | $\forall x \forall y (x' = y' \rightarrow x = y)$ | Ass |
| 9 | (10) | $\forall y (0' = y' \rightarrow 0 = y)$ | UE, 9 |
| 9 | (11) | $(0' = 0'' \rightarrow 0 = 0')$ | UE, 10 |
| 1, 2, 3, 9 | (12) | $0 = 0'$ | Taut, 8, 11 |
| 13 | (13) | $\forall x \neg 0 = x'$ | Ass |
| 13 | (14) | $\neg 0 = 0'$ | UE, 13 |
| 1, 2, 3, 9, 13 | (15) | $(0 = 0' \& \neg 0 = 0')$ | Taut, 12, 14 |
| 1, 2, 9, 13 | (16) | $((0' + 0') = 0' \rightarrow (0 = 0' \& \neg 0 = 0'))$ | CP, 15 |
| 1, 2, 9, 13 | (17) | $\neg (0' + 0') = 0'$ | Taut, 16 |

As assumptions 1, 2, 9 and 13 are axioms Q4, Q5, Q1 and Q2 respectively, we have shown that $\vdash_Q (1 + 1) \neq 1$.

Notice that, at line 8, by legitimate use of the Sub Rule we could have derived either $0' = 0''$ (as we did) or $0'' = 0'$. We chose to derive $0' = 0''$ as previous experience tells us that from this we can derive $0 = 0'$ which, along with the consequence $\neg 0 = 0'$ of axiom Q2, gives us the contradiction that we are seeking.

Solution 1.6

Our aim is to derive the formula $\exists y(x' + 0) = y'$ and then use the UI Rule. As $\exists y(x' + 0) = y'$ is an existential formula, we shall first try to derive $(x' + 0) = \tau'$ for some term τ . The term τ we choose should ideally be such that $(x' + 0) = \tau'$ is true in all interpretations of Q , in particular in the standard interpretation $\mathcal{N}$. Therefore we need τ such that, for any value in $\mathbb{N}$ given to x , $(x' + 0) = \tau'$ is true in $\mathcal{N}$. The obvious choice for τ is the term x . Thus in our proof we first derive $(x' + 0) = x'$.

- 1 (1) $\forall x(x + 0) = x$ Ass
- 1 (2) $(x' + 0) = x'$ UE, 1
- 1 (3) $\exists y(x' + 0) = y'$ EI, 2
- 1 (4) $\forall x \exists y(x' + 0) = y'$ UI, 3

As assumption 1 is axiom Q4, we have shown that $\vdash_Q \forall x \exists y(x' + 0) = y'$.

Solution 1.7

- 1 (1) $\forall x(x + 0) = x$ Ass
- 2 (2) $\forall x(x \cdot 0) = 0$ Ass
- (3) $(0 \cdot (x + 0)) = (0 \cdot (x + 0))$ II
- 1 (4) $(x + 0) = x$ UE, 1
- 1 (5) $(0 \cdot (x + 0)) = (0 \cdot x)$ Sub, 3, 4
- (6) $((0 \cdot x) + (0 \cdot 0)) = ((0 \cdot x) + (0 \cdot 0))$ II
- 2 (7) $(0 \cdot 0) = 0$ UE, 2
- 2 (8) $((0 \cdot x) + 0) = ((0 \cdot x) + (0 \cdot 0))$ Sub, 6, 7
- 1 (9) $((0 \cdot x) + 0) = (0 \cdot x)$ UE, 1
- 1,2 (10) $(0 \cdot x) = ((0 \cdot x) + (0 \cdot 0))$ Sub, 8, 9
- 1,2 (11) $(0 \cdot (x + 0)) = ((0 \cdot x) + (0 \cdot 0))$ Sub, 5, 10
- 1,2 (12) $\forall x(0 \cdot (x + 0)) = ((0 \cdot x) + (0 \cdot 0))$ UI, 11

As assumptions 1 and 2 are axioms Q4 and Q6, we have shown that $\vdash_Q \forall x(0 \cdot (x + 0)) = ((0 \cdot x) + (0 \cdot 0))$.

Solution 1.8

Following technique (T10), we aim first to derive $(\tau' \cdot \tau) = (\tau + \mathbf{0}')$ for some term τ and then use the EI Rule. As the only solution in the natural numbers to $(y + 1) \cdot y = y + 1$ is $y = 1$, the obvious choice for τ is the term $\mathbf{0}'$. So we shall try to derive $(\mathbf{0}'' \cdot \mathbf{0}') = (\mathbf{0}' + \mathbf{0}')$. We shall simplify both $(\mathbf{0}'' \cdot \mathbf{0}')$ and $(\mathbf{0}' + \mathbf{0}')$ to $\mathbf{0}''$ and then use the Sub Rule to obtain $(\mathbf{0}'' \cdot \mathbf{0}') = (\mathbf{0}' + \mathbf{0}')$.

1	(1)	$\forall x (x + \mathbf{0}) = x$	Ass
2	(2)	$\forall x \forall y (x + y') = (x + y)'$	Ass
3	(3)	$\forall x (x \cdot \mathbf{0}) = \mathbf{0}$	Ass
4	(4)	$\forall x \forall y (x \cdot y') = ((x \cdot y) + x)$	Ass
	(5)	$(\mathbf{0}'' \cdot \mathbf{0}') = (\mathbf{0}'' \cdot \mathbf{0}')$	II
4	(6)	$\forall y (\mathbf{0}'' \cdot y') = ((\mathbf{0}'' \cdot y) + \mathbf{0}'')$	UE, 4
4	(7)	$(\mathbf{0}'' \cdot \mathbf{0}') = ((\mathbf{0}'' \cdot \mathbf{0}) + \mathbf{0}'')$	UE, 6
3	(8)	$(\mathbf{0}'' \cdot \mathbf{0}) = \mathbf{0}$	UE, 3
3, 4	(9)	$(\mathbf{0}'' \cdot \mathbf{0}') = (\mathbf{0} + \mathbf{0}'')$	Sub, 7, 8
2	(10)	$\forall y (\mathbf{0} + y') = (\mathbf{0} + y)'$	UE, 2
2	(11)	$(\mathbf{0} + \mathbf{0}'') = (\mathbf{0} + \mathbf{0}')'$	UE, 10
2	(12)	$(\mathbf{0} + \mathbf{0}') = (\mathbf{0} + \mathbf{0})'$	UE, 10
1	(13)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, 1
1, 2	(14)	$(\mathbf{0} + \mathbf{0}') = \mathbf{0}'$	Sub, 12, 13
1, 2	(15)	$(\mathbf{0} + \mathbf{0}'') = \mathbf{0}''$	Sub, 11, 14
1, 2, 3, 4	(16)	$(\mathbf{0}'' \cdot \mathbf{0}') = \mathbf{0}''$	Sub, 9, 15
	(17)	$(\mathbf{0}' + \mathbf{0}') = (\mathbf{0}' + \mathbf{0}')$	II
2	(18)	$\forall y (\mathbf{0}' + y') = (\mathbf{0}' + y)'$	UE, 2
2	(19)	$(\mathbf{0}' + \mathbf{0}') = (\mathbf{0}' + \mathbf{0})'$	UE, 18
1	(20)	$(\mathbf{0}' + \mathbf{0}) = \mathbf{0}'$	UE, 1
1, 2	(21)	$(\mathbf{0}' + \mathbf{0}') = \mathbf{0}'$	Sub, 19, 20
1, 2	(22)	$\mathbf{0}'' = (\mathbf{0}' + \mathbf{0}')$	Sub, 17, 21
1, 2, 3, 4	(23)	$(\mathbf{0}'' \cdot \mathbf{0}') = (\mathbf{0}' + \mathbf{0}')$	Sub, 16, 22
1, 2, 3, 4	(24)	$\exists y (y' \cdot y) = (y + \mathbf{0}')$	EI, 23

As assumptions 1, 2, 3 and 4 are axioms Q4, Q5, Q6 and Q7, we have shown that $\vdash_Q \exists y (y' \cdot y) = (y + \mathbf{0}')$.

Solution 1.9

We adopt the same strategy as in Example 1.8.

1	(1)	$\forall x \forall y (x \cdot y') = ((x \cdot y) + x)$	Ass
1	(2)	$\forall y (z \cdot y') = ((z \cdot y) + z)$	UE, 1
1	(3)	$(z \cdot x') = ((z \cdot x) + z)$	UE, 2
1	(4)	$\forall z (z \cdot x') = ((z \cdot x) + z)$	UI, 3
1	(5)	$(y \cdot x') = ((y \cdot x) + y)$	UE, 4
1	(6)	$\forall x (y \cdot x') = ((y \cdot x) + y)$	UI, 5
1	(7)	$\forall y \forall x (y \cdot x') = ((y \cdot x) + y)$	UI, 6

As assumption 1 is axiom Q7, we have shown that $\vdash_Q \forall y \forall x (y \cdot x') = ((y \cdot x) + y)$.

Solution 1.10

- (a) This sentence is a theorem of Q , as is shown by the following formal proof.

1	(1)	$\forall x (x + 0) = x$	Ass
2	(2)	$\forall x \forall y (x + y') = (x + y)'$	Ass
1	(3)	$(x + 0) = x$	UE, 1
2	(4)	$\forall y (x + y') = (x + y)'$	UE, 2
2	(5)	$(x + 0') = (x + 0)'$	UE, 4
1, 2	(6)	$(x + 0') = x'$	Sub, 3, 5
1, 2	(7)	$\forall x (x + 0') = x'$	UI, 6

As assumptions 1 and 2 are the axioms $Q4$ and $Q5$, we have shown that $\vdash_Q \forall x (x + 0') = x'$.

- (b) This sentence is not a theorem of Q . In the interpretation $\mathcal{N}^{**}$ given in Example 3.6 of *Unit 4*, we have $\alpha \cdot 0' = \alpha \cdot 1 = \beta$, and so $\alpha \cdot 0' \neq \alpha$. Hence the sentence $\forall x (x \cdot 0') = x$ is not true in the interpretation $\mathcal{N}^{**}$. As all the theorems of Q are true in this interpretation, it follows by the Correctness Theorem that this sentence is not a theorem of Q .

Solution 2.1

Let i and j be natural numbers and suppose that $i + j = k$. Then also $j + i = k$. Hence, by Theorem 2.2(a), we have both

$$\vdash_Q (i + j) = k \quad \text{and} \quad \vdash_Q (j + i) = k \quad (\text{S.1})$$

Now it is quite straightforward to show that

$$(i + j) = k, (j + i) = k \vdash (i + j) = (j + i)$$

All we have to do is to be careful about the use of the Sub Rule:

1	(1)	$(i + j) = k$	Ass
2	(2)	$(j + i) = k$	Ass
	(3)	$(j + i) = (j + i)$	II
2	(4)	$k = (j + i)$	Sub, 2, 3
1, 2	(5)	$(i + j) = (j + i)$	Sub, 1, 4

Hence it follows from (S.1) that $\vdash_Q (i + j) = (j + i)$.

Solution 2.2

Let i and j be natural numbers and suppose that $i \times j = k$. Then also $j \times i = k$. Hence, by Theorem 2.3(a),

$$\vdash_Q (i \cdot j) = k \quad \text{and} \quad \vdash_Q (j \cdot i) = k$$

In almost identical manner to that in Solution 2.1, we can show that

$$(i \cdot j) = k, (j \cdot i) = k \vdash (i \cdot j) = (j \cdot i)$$

Hence $\vdash_Q (i \cdot j) = (j \cdot i)$.

Solution 2.3

- (a) $\phi(3, 4, 5)$ is the formula $((3 \cdot 4) = 5 \vee 5 = 4)$.

$\phi(1, 2, x_3)$ is the formula $((1 \cdot 2) = x_3 \vee x_3 = 2)$.

- (b) $\phi(1, 2, 3, 4)$ is the formula $(\exists x_4 (2 \cdot (1 \cdot 3)) = x_4 \rightarrow 4 = 3)$.

$\phi(1, 2, 3, x_4)$ is the formula $(\exists x_4 (2 \cdot (1 \cdot 3)) = x_4 \rightarrow x_4 = 3)$.

Only the third occurrence of x_4 in ϕ is a free occurrence.

Solution 2.4

We show that the formula $x_2 = x'_1$ represents the successor function in Q .

$x_2 = x'_1$ is not the only formula that does this, but it is the most obvious choice.

Let q and k be natural numbers such that $\text{succ}(q) = k$. We need to show that:

- (a) $\vdash_Q \mathbf{k} = \mathbf{q}'$
- (b) $\vdash_Q \forall x_2 (x_2 = \mathbf{q}' \rightarrow x_2 = \mathbf{k})$

Now, as $\text{succ}(q) = k$, we have $k = q + 1$ and hence the term $\mathbf{k}$ is identical to the term $\mathbf{q}'$. Thus (a) is equivalent to

- (a') $\vdash_Q \mathbf{q}' = \mathbf{q}'$

and (b) is equivalent to

- (b') $\vdash_Q \forall x_2 (x_2 = \mathbf{q}' \rightarrow x_2 = \mathbf{q}')$.

It is easy to see that (a') and (b') can be proved in exactly the same way as we proved (a) and (b) in Example 2.6. All that needs to be done is to replace the term $\mathbf{0}$ in the formal proofs given in Example 2.6 by the term $\mathbf{q}'$.

Hence we can deduce that the formula $x_2 = x'_1$ represents the successor function in Q .

Solution 2.5

We show that the formula $x_{n+1} = x_m$ represents the projection function U_m^n in Q .

Let $q_1, q_2, \dots, q_n, k$ be natural numbers such that $U_m^n(q_1, q_2, \dots, q_n) = k$. We need to show that:

- (a) $\vdash_Q \mathbf{k} = \mathbf{q}_m$
- (b) $\vdash_Q \forall x_{n+1} (x_{n+1} = \mathbf{q}_m \rightarrow x_{n+1} = \mathbf{k})$

Now, as $U_m^n(q_1, q_2, \dots, q_n) = k$, we have that $k = q_m$. Thus (a) and (b) are equivalent to:

- (a') $\vdash_Q \mathbf{q}_m = \mathbf{q}_m$
- (b') $\vdash_Q \forall x_{n+1} (x_{n+1} = \mathbf{q}_m \rightarrow x_{n+1} = \mathbf{q}_m)$

We can establish the truth of (a') and (b') by giving formal proofs which are again almost identical to those given in Example 2.6: replace the term $\mathbf{0}$ by $\mathbf{q}_m$ and, if $n \neq 1$, replace the variable x_2 by x_{n+1} .

Hence it follows that the formula $x_{n+1} = x_m$ represents the function U_m^n in Q .

Solution 2.6

Although the formal proof below is rather long, it is not as complicated as it looks.

We adopt the standard strategy of first introducing χ_1 , χ_2 and χ_3 as assumptions on lines 1 to 3. Since they are all universal formulas, we expect that we shall need to apply the UE Rule, but we postpone doing this until it becomes clearer what we should substitute for x . So we turn our attention to our desired conclusion. Since this is $\forall x (\theta(\mathbf{q}, x) \rightarrow x = \mathbf{k})$, we aim to derive $(\theta(\mathbf{q}, x) \rightarrow x = \mathbf{k})$ and then use the UI Rule. The standard strategy for deriving the implication $(\theta(\mathbf{q}, x) \rightarrow x = \mathbf{k})$ is to assume $\theta(\mathbf{q}, x)$ and to attempt to derive $x = \mathbf{k}$. So on line 4 we introduce $\theta(\mathbf{q}, x)$ as an assumption. Since this assumption is an existential formula, we follow our technique (T3) (Unit 6, Subsection 1.2) and introduce as assumptions on lines 5 and 6 the formulas obtained from $\theta(\mathbf{q}, x)$ by successively dropping the initial existential quantifiers.

We now focus our attention on deriving $x = \mathbf{k}$. We see from line 3 that we could deduce this if we could first derive $\psi(\mathbf{q}_1, \mathbf{q}_2, x)$. Now from the conjunction on line 6 we can derive $\psi(y_1, y_2, x)$. So all we need now is to be able to derive $y_1 = \mathbf{q}_1$ and $y_2 = \mathbf{q}_2$ so that we can then derive $\psi(\mathbf{q}_1, \mathbf{q}_2, x)$ using two applications of the Sub Rule. Plainly using assumption 1 we can derive $(\phi_1(\mathbf{q}, y_1) \rightarrow y_1 = \mathbf{q}_1)$ and since $\phi_1(\mathbf{q}, y_1)$ is one of the conjuncts making up the formula on line 6 we can thus derive $y_1 = \mathbf{q}_1$. In a similar way we can derive $y_2 = \mathbf{q}_2$.

Thus we are led to the following formal proof.

1	(1)	$\forall x (\phi_1(\mathbf{q}, x) \rightarrow x = \mathbf{q}_1)$	Ass
2	(2)	$\forall x (\phi_2(\mathbf{q}, x) \rightarrow x = \mathbf{q}_2)$	Ass
3	(3)	$\forall x (\psi(\mathbf{q}_1, \mathbf{q}_2, x) \rightarrow x = \mathbf{k})$	Ass
4	(4)	$\exists y_1 \exists y_2 ((\phi_1(\mathbf{q}, y_1) \& \phi_2(\mathbf{q}, y_2)) \& \psi(y_1, y_2, x))$	Ass
5	(5)	$\exists y_2 ((\phi_1(\mathbf{q}, y_1) \& \phi_2(\mathbf{q}, y_2)) \& \psi(y_1, y_2, x))$	Ass
6	(6)	$((\phi_1(\mathbf{q}, y_1) \& \phi_2(\mathbf{q}, y_2)) \& \psi(y_1, y_2, x))$	Ass
6	(7)	$\phi_1(\mathbf{q}, y_1)$	Taut, 6
1	(8)	$(\phi_1(\mathbf{q}, y_1) \rightarrow y_1 = \mathbf{q}_1)$	UE, 1
1, 6	(9)	$y_1 = \mathbf{q}_1$	Taut, 7, 8
6	(10)	$\phi_2(\mathbf{q}, y_2)$	Taut, 6
2	(11)	$(\phi_2(\mathbf{q}, y_2) \rightarrow y_2 = \mathbf{q}_2)$	UE, 2
2, 6	(12)	$y_2 = \mathbf{q}_2$	Taut, 10, 11
6	(13)	$\psi(y_1, y_2, x)$	Taut, 6
1, 6	(14)	$\psi(\mathbf{q}_1, y_2, x)$	Sub, 9, 13
1, 2, 6	(15)	$\psi(\mathbf{q}_1, \mathbf{q}_2, x)$	Sub, 12, 14
3	(16)	$(\psi(\mathbf{q}_1, \mathbf{q}_2, x) \rightarrow x = \mathbf{k})$	UE, 3
1, 2, 3, 6	(17)	$x = \mathbf{k}$	Taut, 15, 16
1, 2, 3, 5	(18)	$x = \mathbf{k}$	EH, 17
1, 2, 3, 4	(19)	$x = \mathbf{k}$	EH, 18
1, 2, 3	(20)	$(\exists y_1 \exists y_2 ((\phi_1(\mathbf{q}, y_1) \& \phi_2(\mathbf{q}, y_2)) \& \psi(y_1, y_2, x)) \rightarrow x = \mathbf{k})$	CP, 19
1, 2, 3	(21)	$\forall x (\exists y_1 \exists y_2 ((\phi_1(\mathbf{q}, y_1) \& \phi_2(\mathbf{q}, y_2)) \& \psi(y_1, y_2, x)) \rightarrow x = \mathbf{k})$	UI, 20

Solution 3.1

$\neg\phi$ is the formula

$$\neg(\exists y x = (2 \cdot y) \& \neg\exists y x = (4 \cdot y))$$

Hence $D_{\neg\phi}$ is the formula

$$\exists x (x = m \& \neg(\exists y x = (2 \cdot y) \& \neg\exists y x = (4 \cdot y)))$$

where m is the Gödel number of $\neg\phi$.

$\neg\phi$ expresses the property 'it is not the case that x is divisible by 2 but is not divisible by 4' which is equivalent to 'either x is not divisible by 2 or it is divisible by 4'. Now m , the Gödel number of $\neg\phi$, is given by $m = 2^7 3^1 \dots$ and hence is divisible by 4. So $D_{\neg\phi}$ is true under the standard interpretation.

Solution 3.2

The strings consisting of the single symbols $(, \&,)$ have Gödel numbers $2^1, 2^3, 2^2$ respectively. Now $(\phi \& \psi)$ is the string obtained by writing the strings $(, \phi, \&, \psi,)$ one after the other. These strings have Gödel numbers $2, m, 2^3, n, 2^2$ respectively. Hence

$$\text{conj}(m, n) = (((2 * m) * 2^3) * n) * 2^2.$$

Since the function $*$ is primitive recursive, so also is the function conj .

Solution 3.3

The strings consisting of the single symbols **0** and **'** have Gödel numbers 2^{10} and 2^{11} , respectively. Hence the function num can be defined by the primitive recursion

$$\begin{aligned}\text{num}(0) &= 2^{10}, \\ \text{num}(n+1) &= \text{num}(n) * 2^{11},\end{aligned}$$

since $\mathbf{n} + 1$ is the string $\mathbf{n}$ followed by the string **'**. Hence, as $*$ is a primitive recursive function, so also is the function num.

Alternatively, since $\mathbf{n}$ consists of the $n + 1$ symbols comprising **0** followed by n copies of **'**, we can see directly that

$$\text{num}(n) = 2^{10} \times \prod_{i=1}^n p_{i+1}^{11}.$$

Since multiplication, exponentiation, addition and the function $n \mapsto p_n$ are primitive recursive and since primitive recursive functions are closed under bounded products (Theorem 3.2 of *Unit 2*), it follows that num is primitive recursive.

SOLUTIONS TO ADDITIONAL EXERCISES

Section 1

- 1 (a) 1 (1) $\forall x (x + \mathbf{0}) = x$ Ass
1 (2) $(y' + \mathbf{0}) = y'$ UE, 1
1 (3) $\forall y (y' + \mathbf{0}) = y'$ UI, 2
1 (4) $\exists x \forall y (y' + x) = y'$ EI, 3

As assumption 1 is axiom Q4, we have shown that
 $\vdash_Q \exists x \forall y (y' + x) = y'$.

- (b) 1 (1) $\forall x (x + \mathbf{0}) = x$ Ass
2 (2) $\forall x (x \cdot \mathbf{0}) = \mathbf{0}$ Ass
(3) $(\mathbf{0} + (x \cdot \mathbf{0})) = (\mathbf{0} + (x \cdot \mathbf{0}))$ II
2 (4) $(x \cdot \mathbf{0}) = \mathbf{0}$ UE, 2
2 (5) $(\mathbf{0} + (x \cdot \mathbf{0})) = (\mathbf{0} + \mathbf{0})$ Sub, 3, 4
1 (6) $(\mathbf{0} + \mathbf{0}) = \mathbf{0}$ UE, 1
1, 2 (7) $(\mathbf{0} + (x \cdot \mathbf{0})) = \mathbf{0}$ Sub, 5, 6
(8) $((\mathbf{0} + x) \cdot \mathbf{0}) = ((\mathbf{0} + x) \cdot \mathbf{0})$ II
2 (9) $((\mathbf{0} + x) \cdot \mathbf{0}) = \mathbf{0}$ UE, 2
2 (10) $\mathbf{0} = ((\mathbf{0} + x) \cdot \mathbf{0})$ Sub, 8, 9
1, 2 (11) $(\mathbf{0} + (x \cdot \mathbf{0})) = ((\mathbf{0} + x) \cdot \mathbf{0})$ Sub, 7, 10
1, 2 (12) $\forall x (\mathbf{0} + (x \cdot \mathbf{0})) = ((\mathbf{0} + x) \cdot \mathbf{0})$ UI, 11

As assumptions 1 and 2 are axioms Q4 and Q6, we have shown that
 $\vdash_Q \forall x (\mathbf{0} + (x \cdot \mathbf{0})) = ((\mathbf{0} + x) \cdot \mathbf{0})$.

- (c)
- | | | | |
|------|-----|--------------------------------------|-----------|
| 1 | (1) | $\forall x \neg 0 = x'$ | Ass |
| 2 | (2) | $\forall x (x + 0) = x$ | Ass |
| 1 | (3) | $\neg 0 = (0 + 0)''$ | UE, 1 |
| 2 | (4) | $(0 + 0) = 0$ | UE, 2 |
| | (5) | $(0 + 0) = (0 + 0)$ | II |
| 2 | (6) | $0 = (0 + 0)$ | Sub, 4, 5 |
| 1, 2 | (7) | $\neg (0 + 0) = (0 + 0)''$ | Sub, 3, 6 |
| 1, 2 | (8) | $\exists x \neg (x + x) = (x + x)''$ | EI, 7 |

As assumptions 1 and 2 are axioms $Q2$ and $Q4$, we have shown that $\vdash_Q \exists x \neg (x + x) = (x + x)''$.

- (d)
- | | | | |
|------|------|---|-----------|
| 1 | (1) | $\forall x (x + 0) = x$ | Ass |
| 2 | (2) | $\forall x \forall y (x + y') = (x + y)'$ | Ass |
| 2 | (3) | $\forall y (x + y') = (x + y)'$ | UE, 2 |
| 2 | (4) | $(x + 0'') = (x + 0')'$ | UE, 3 |
| 2 | (5) | $(x + 0') = (x + 0)'$ | UE, 3 |
| 1 | (6) | $(x + 0) = x$ | UE, 1 |
| 1, 2 | (7) | $(x + 0') = x'$ | Sub, 5, 6 |
| 1, 2 | (8) | $(x + 0'') = x''$ | Sub, 4, 7 |
| 1, 2 | (9) | $\exists y (x + y) = x''$ | EI, 8 |
| 1, 2 | (10) | $\forall x \exists y (x + y) = x''$ | UI, 9 |

As assumptions 1 and 2 are axioms $Q4$ and $Q5$, we have shown that $\vdash_Q \forall x \exists y (x + y) = x''$.

- 2 (a) This sentence is a theorem of Q , as is shown by the following formal proof.

- | | | | |
|------|------|---|------------|
| 1 | (1) | $\forall x (x + 0) = x$ | Ass |
| 2 | (2) | $\forall x (x \cdot 0) = 0$ | Ass |
| | (3) | $(x \cdot (0' + 0)) = (x \cdot (0' + 0))$ | II |
| 1 | (4) | $(0' + 0) = 0'$ | UE, 1 |
| 1 | (5) | $(x \cdot (0' + 0)) = (x \cdot 0')$ | Sub, 3, 4 |
| | (6) | $((x \cdot 0') + (x \cdot 0)) = ((x \cdot 0') + (x \cdot 0))$ | II |
| 2 | (7) | $(x \cdot 0) = 0$ | UE, 2 |
| 2 | (8) | $((x \cdot 0') + 0) = ((x \cdot 0') + (x \cdot 0))$ | Sub, 6, 7 |
| 1 | (9) | $((x \cdot 0') + 0) = (x \cdot 0')$ | UE, 1 |
| 1, 2 | (10) | $((x \cdot 0') = ((x \cdot 0') + (x \cdot 0))$ | Sub, 8, 9 |
| 1, 2 | (11) | $(x \cdot (0' + 0)) = ((x \cdot 0') + (x \cdot 0))$ | Sub, 5, 10 |
| 1, 2 | (12) | $\forall x (x \cdot (0' + 0)) = ((x \cdot 0') + (x \cdot 0))$ | UI, 11 |

As assumptions 1 and 2 are axioms $Q4$ and $Q6$, we have shown that $\vdash_Q \forall x (x \cdot (0' + 0)) = ((x \cdot 0') + (x \cdot 0))$.

- (b) This sentence is not a theorem of Q . In the interpretation $\mathcal{N}^{**}$ given in Example 3.6 of *Unit 4*, interpret x as α . We have

$$(0' + 0) \cdot \alpha = 1 \cdot \alpha = \alpha,$$

whereas

$$(0' \cdot \alpha) + (0 \cdot \alpha) = (1 \cdot \alpha) + (0 \cdot \alpha) = \alpha + \alpha = \beta,$$

so that $(0' + 0) \cdot \alpha \neq (0' \cdot \alpha) + (0 \cdot \alpha)$.

So the sentence $\forall x ((0' + 0) \cdot x) = ((0' \cdot x) + (0 \cdot x))$ is not true in $\mathcal{N}^{**}$. As all the theorems of Q are true in this interpretation, it follows by the Correctness Theorem that this sentence is not a theorem of Q .

Note that we could have reduced the term $(x \cdot 0')$ to $((x \cdot 0) + x)$ which can be reduced further to $(0 + x)$. While not incorrect, it would have served only to lengthen the proof. Note also that the term $(0 + x)$ cannot be reduced any further using the axioms of Q .

Interpreting x as β also provides a suitable counter-example.

- 3 There is just one point we need to be careful about. We need to cover the possibility that u is the same as y . To deal with this possibility we choose z to be a variable which is distinct from both y and v . Then the following is always a valid formal proof.

- 1 (1) $\forall x \forall y (x + y)' = (x + y)'$ Ass
- 1 (2) $\forall y (z + y)' = (z + y)'$ UE, 1
- 1 (3) $(z + v)' = (z + v)'$ UE, 2
- 1 (4) $\forall z (z + v)' = (z + v)'$ UI, 3
- 1 (5) $(u + v)' = (u + v)'$ UE, 4
- 1 (6) $\forall v (u + v)' = (u + v)'$ UI, 5
- 1 (7) $\forall u \forall v (u + v)' = (u + v)'$ UI, 6

If u is distinct from y , we can take z to be u and lines 3 and 4 may then be omitted; but if u is identical to y , it is not a valid use of the UE Rule to derive $\forall y (u + y)' = (u + y)'$ from assumption 1.

As assumption 1 is axiom Q4, we have shown that $\vdash_Q \forall u \forall v (u + v)' = (u + v)'$.

Section 2

- 1 We give a proof by Mathematical Induction that, for all $i \geq 0$

$$\vdash_Q \forall x (x' + i) = (x + i')$$

For the basis of the induction we need to establish this result for the case $i = 0$. That is, we need to show that

$$\vdash_Q \forall x (x' + 0) = (x + 0')$$

This is done by means of the following formal proof.

- 1 (1) $\forall x (x + 0) = x$ Ass
- 2 (2) $\forall x \forall y (x + y)' = (x + y)'$ Ass
- 2 (3) $\forall y (x + y)' = (x + y)'$ UE, 2
- 2 (4) $(x + 0)' = (x + 0)'$ UE, 3
- 1 (5) $(x + 0) = x$ UE, 1
- 1, 2 (6) $(x + 0)' = x'$ Sub, 4, 5
- 1 (7) $(x' + 0) = x'$ UE, 1
- (8) $(x + 0)' = (x + 0)'$ II
- 1, 2 (9) $x' = (x + 0)'$ Sub, 6, 8
- 1, 2 (10) $(x' + 0) = (x + 0)'$ Sub, 7, 9
- 1, 2 (11) $\forall x (x' + 0) = (x + 0)'$ UI, 10

Since assumptions 1 and 2 are axioms Q4 and Q5, it follows that $\vdash_Q \forall x (x' + 0) = (x + 0)'$.

Next, for the induction step, we need to show that, for each natural number n ,

$$\vdash_Q \forall x (x' + n) = (x + n')$$

implies

$$\vdash_Q \forall x (x' + n') = (x + n'')$$

We achieve this by the following formal proof.

- 1 (1) $\forall x (x' + \mathbf{n}) = (x + \mathbf{n}')$ Ass
- 2 (2) $\forall x \forall y (x + y)' = (x + y)'$ Ass
- 2 (3) $\forall y (x' + y') = (x' + y)'$ UE, 2
- 2 (4) $(x' + \mathbf{n}') = (x' + \mathbf{n})'$ UE, 3
- 1 (5) $(x' + \mathbf{n}) = (x + \mathbf{n}')$ UE, 1
- 1, 2 (6) $(x' + \mathbf{n}') = (x + \mathbf{n}')'$ Sub, 4, 5
- 2 (7) $\forall y (x + y)' = (x + y)'$ UE, 2
- 2 (8) $(x + \mathbf{n}'') = (x + \mathbf{n}')'$ UE, 7
- (9) $(x + \mathbf{n}'') = (x + \mathbf{n}'')$ II
- 2 (10) $(x + \mathbf{n}')' = (x + \mathbf{n}'')$ Sub, 8, 9
- 1, 2 (11) $(x' + \mathbf{n}') = (x + \mathbf{n}'')$ Sub, 6, 10
- 1, 2 (12) $\forall x (x' + \mathbf{n}') = (x + \mathbf{n}'')$ UI, 11

Assumption 2 is axiom Q5, so this formal proof shows that

$$\forall x (x' + \mathbf{n}) = (x + \mathbf{n}') \vdash_Q \forall x (x' + \mathbf{n}') = (x + \mathbf{n}'')$$

This completes the proof of the induction step.

Hence, using Mathematical Induction, it follows that, for all natural numbers i ,

$$\vdash_Q \forall x (x' + \mathbf{i}) = (x + \mathbf{i}')$$

2 We choose ϕ to be the formula

$$((x_1 = x_2 \ \& \ x_3 = \mathbf{1}) \vee (\neg x_1 = x_2 \ \& \ x_3 = \mathbf{0}))$$

and we show that ϕ represents the function χ_{eq} in Q .

We need to show that, for all natural numbers q_1, q_2 and k with $\chi_{eq}(q_1, q_2) = k$, we have both

$$\vdash_Q \phi(\mathbf{q_1}, \mathbf{q_2}, \mathbf{k}) \quad \text{and} \quad \vdash_Q \forall x_3 (\phi(\mathbf{q_1}, \mathbf{q_2}, x_3) \rightarrow x_3 = \mathbf{k})$$

That is, we need to show that:

- (a) $\vdash_Q ((\mathbf{q_1} = \mathbf{q_2} \ \& \ \mathbf{k} = \mathbf{1}) \vee (\neg \mathbf{q_1} = \mathbf{q_2} \ \& \ \mathbf{k} = \mathbf{0}))$
- (b) $\vdash_Q \forall x_3 (((\mathbf{q_1} = \mathbf{q_2} \ \& \ x_3 = \mathbf{1}) \vee (\neg \mathbf{q_1} = \mathbf{q_2} \ \& \ x_3 = \mathbf{0})) \rightarrow x_3 = \mathbf{k})$

We first consider condition (a). We need to deal with the cases $q_1 = q_2$ and $q_1 \neq q_2$ separately.

Suppose first that $q_1 = q_2$ and hence that $\chi_{eq}(q_1, q_2) = 1$. In this case $k = 1$ and so (a) becomes

$$\vdash_Q ((\mathbf{q_1} = \mathbf{q_1} \ \& \ \mathbf{1} = \mathbf{1}) \vee (\neg \mathbf{q_1} = \mathbf{q_1} \ \& \ \mathbf{1} = \mathbf{0})) \quad (1)$$

Now by the II Rule we have that both $\vdash_Q \mathbf{q_1} = \mathbf{q_1}$ and $\vdash_Q \mathbf{1} = \mathbf{1}$.

Hence, using the Tautology Rule, $\vdash_Q (\mathbf{q_1} = \mathbf{q_1} \ \& \ \mathbf{1} = \mathbf{1})$. Therefore, using the Tautology Rule again with the tautology $(\theta \rightarrow (\theta \vee \psi))$, we have that (1) holds. So (a) holds in this case.

Second, suppose that $q_1 \neq q_2$. So $\chi_{eq}(q_1, q_2) = 0$ and hence $k = 0$. So (a) becomes

$$\vdash_Q ((\mathbf{q_1} = \mathbf{q_2} \ \& \ \mathbf{0} = \mathbf{1}) \vee (\neg \mathbf{q_1} = \mathbf{q_2} \ \& \ \mathbf{0} = \mathbf{0})) \quad (2)$$

Since $q_1 \neq q_2$, we have, by Theorem 2.1(b), $\vdash_Q \neg \mathbf{q_1} = \mathbf{q_2}$, and, by the II Rule, $\vdash_Q \mathbf{0} = \mathbf{0}$. Hence, as in the previous case, but this time using the tautology $(\psi \rightarrow (\theta \vee \psi))$, we see that (2) holds. Hence (a) holds also in this case.

Hence we have shown that (a) holds for all natural numbers q_1, q_2 and k with $\chi_{eq}(q_1, q_2) = k$.

Next we consider condition (b). We again need to deal with the cases $q_1 = q_2$ and $q_1 \neq q_2$ separately.

We suppose first that $q_1 = q_2$. So, as before, $k = 1$, and (b) becomes

$$\vdash_Q \forall x_3 (((\mathbf{q}_1 = \mathbf{q}_1 \& x_3 = 1) \vee (\neg \mathbf{q}_1 = \mathbf{q}_1 \& x_3 = 0)) \rightarrow x_3 = 1)$$

This is proved by the following formal proof.

- | | | |
|-----|---|---------|
| (1) | $\mathbf{q}_1 = \mathbf{q}_1$ | II |
| (2) | $((\mathbf{q}_1 = \mathbf{q}_1 \& x_3 = 1) \vee (\neg \mathbf{q}_1 = \mathbf{q}_1 \& x_3 = 0)) \rightarrow x_3 = 1$ | Taut, 1 |
| (3) | $\forall x_3 (((\mathbf{q}_1 = \mathbf{q}_1 \& x_3 = 1) \vee (\neg \mathbf{q}_1 = \mathbf{q}_1 \& x_3 = 0)) \rightarrow x_3 = 1)$ | UE, 2 |

On line 2, the use of the Tautology Rule is justified because the formula

$$(\theta \rightarrow (((\theta \& \psi) \vee (\neg \theta \& \chi)) \rightarrow \psi))$$

is a tautology, as can easily be checked; here θ is $\mathbf{q}_1 = \mathbf{q}_1$, ψ is $x_3 = 1$ and χ is $x_3 = 0$. Note that we have derived the desired result without having to use any of the axioms of Q .

The case where $q_1 \neq q_2$ is very similar. In this case $k = 0$, and, by Theorem 2.1(b), $\vdash_Q \neg \mathbf{q}_1 = \mathbf{q}_2$. It then follows by use of the Tautology Rule with the tautology

$$(\neg \theta \rightarrow (((\theta \& \psi) \vee (\neg \theta \& \chi)) \rightarrow \chi))$$

with θ , ψ and χ as before, that

$$\vdash_Q (((\mathbf{q}_1 = \mathbf{q}_2 \& x_3 = 1) \vee (\neg \mathbf{q}_1 = \mathbf{q}_2 \& x_3 = 0)) \rightarrow x_3 = 0)$$

Hence, using the UI Rule, we have

$$\vdash_Q \forall x_3 (((\mathbf{q}_1 = \mathbf{q}_2 \& x_3 = 1) \vee (\neg \mathbf{q}_1 = \mathbf{q}_2 \& x_3 = 0)) \rightarrow x_3 = 0)$$

which shows that (b) holds also in this case.

It follows that the formula ϕ represents the function χ_{eq} in Q .

- 3 (a) Suppose $q_1, q_2, \dots, q_n, j$ and k are natural numbers such that $f(q_1, q_2, \dots, q_n) = k$ and $j \neq k$.

Then since ϕ represents f in T we have

$$\vdash_T \forall x_{n+1} (\phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, x_{n+1}) \rightarrow x_{n+1} = \mathbf{k})$$

Also, as $j \neq k$ and T extends Q , it follows from Theorem 2.1(b) that

$$\vdash_T \neg \mathbf{j} = \mathbf{k}$$

Now consider the following formal proof by contradiction.

- | | | | |
|---------|-----|---|------------|
| 1 | (1) | $\forall x_{n+1} (\phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, x_{n+1}) \rightarrow x_{n+1} = \mathbf{k})$ | Ass |
| 2 | (2) | $\neg \mathbf{j} = \mathbf{k}$ | Ass |
| 3 | (3) | $\phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{j})$ | Ass |
| 1 | (4) | $(\phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{j}) \rightarrow \mathbf{j} = \mathbf{k})$ | UE, 1 |
| 1, 3 | (5) | $\mathbf{j} = \mathbf{k}$ | Taut, 3, 4 |
| 1, 2, 3 | (6) | $(\mathbf{j} = \mathbf{k} \& \neg \mathbf{j} = \mathbf{k})$ | Taut, 2, 5 |
| 1, 2 | (7) | $(\phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{j}) \rightarrow (\mathbf{j} = \mathbf{k} \& \neg \mathbf{j} = \mathbf{k}))$ | CP, 6 |
| 1, 2 | (8) | $\neg \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{j})$ | Taut, 7 |

Since we have seen above that the assumptions 1 and 2 are derivable in T , it follows that

$$\vdash_T \neg \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{j})$$

as required.

- (b) Suppose $q_1, q_2, \dots, q_n$ and k are natural numbers such that $f(q_1, q_2, \dots, q_n) = k$. Then as ϕ represents f in T we have

$$\vdash_T \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k})$$

Thus

$$f(q_1, q_2, \dots, q_n) = k \implies \vdash_T \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k}) \quad (1)$$

Now suppose $f(q_1, q_2, \dots, q_n) \neq k$. Then, by (a),

$$\vdash_T \neg \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k})$$

Hence as T is consistent, it must be that $\phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k})$ is not derivable in T . So we have

$$\vdash_T \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k}) \implies f(q_1, q_2, \dots, q_n) = k. \quad (2)$$

By (1) and (2)

$$\vdash_T \phi(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n, \mathbf{k}) \iff f(q_1, q_2, \dots, q_n) = k$$

as required.

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